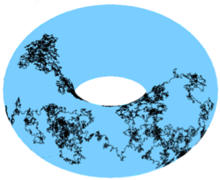
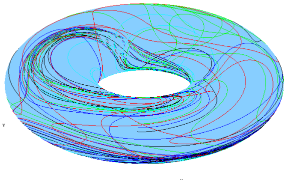
Kinetic Brownian motion in the diffeomorphism group of a closed Riemannian manifold





Joint works with J. Angst, C. Tardif and P. Perruchaud (Rennes)

1. Kinetic Brownian motion in $\mathbb{R}^d$

- Definition. **Kinetic Brownian motion** $(x_t, \dot{x}_t)$ **in** $\mathbb{R}^d$ is the hypoelliptic diffusion with state space $\mathbb{R}^d \times \mathbb{S}^{d-1}$

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dx_t &= \dot{x}_t \, dt, \\
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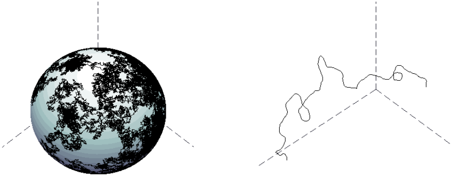

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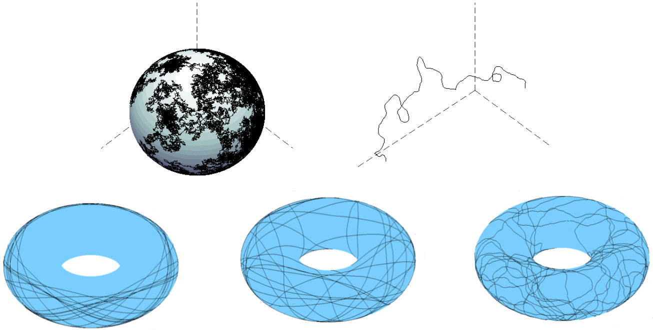

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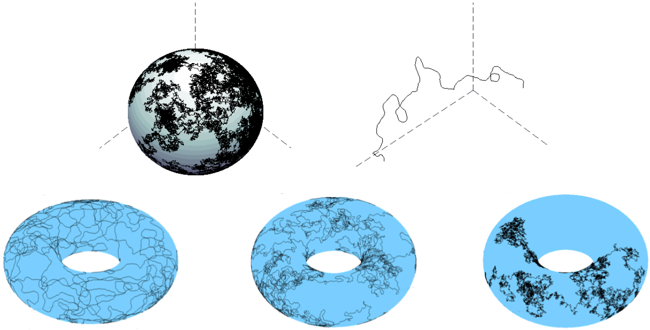

1. Kinetic Brownian motion in $\mathbb{R}^d$ – a hypoelliptic diffusion

- Even in this simple situation, no general result on heat kernel estimates is available.

Perruchaud proved in this PhD thesis an asymptotics in terms of the heat kernel $\overline{p}_t$ of a model *non-Gaussian* diffusion, with an explicit kernel, in a 2-dimensional setting

$$T^1\mathbb{T}^2 \simeq \mathbb{T}^2 \times \mathbb{S}^1 = \big\{(z,\theta)\big\} = \big\{((x,y),\theta)\big\}.$$

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► Theorem (Perruchaud '19) – **Heat kernel estimate.** *Let D be a domain of $T^1\mathbb{T}^2$ where $\overline{p}_1\big((0,0),\bullet + ((1,0),0)\big)$ is bounded away from 0. Then*

$$p_t\big((0,0),((x+t,y),\theta)\big) = \overline{p}_t\big((0,0),((x+t,y),\theta)\big)\,(1+O(t)),$$

uniformly in $\left(t^{-2}x, t^{-3/2}y, t^{-1/2}\theta\right) \in D$.

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• A. Drouot 17' and H.F. Smith 20' provide hypoelliptic regularity estimates and a parametrix for the generator of kinetic Brownian motion.

1. Kinetic Brownian motion in $\mathbb{R}^d$

► Theorem – **Homogenization.** *The time-rescaled position process* $(x_{\sigma^2 t})_{0\leq t\leq 1}$ *converges weakly to a Euclidean Brownian motion with generator* $\frac{4}{d(d-1)}\Delta$, *as* $\sigma \uparrow \infty$.

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Idea of proof. The dynamics of kinetic Brownian motion is given by the SDE

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\begin{aligned}
dx_t^i &= \dot{x}_t^i \, dt \\
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Set $X_t^\sigma := x_{\sigma^2 t}$. Then

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X_t^\sigma = x_0 + \frac{2}{d-1} \, \frac{1}{\sigma^2} \big(\dot{x}_0 - \dot{x}_{\sigma^2 t}\big) + M_t^\sigma,
$$

with

$$
\big\langle M^{\sigma,i}, M^{\sigma,j} \big\rangle_t = \frac{4}{(d-1)^2} \, \frac{1}{\sigma^2} \int_0^{\sigma^2 t} \big(\delta^{ij} - \dot{x}_s^{\sigma,i} \dot{x}_s^{\sigma,j}\big) \, ds.
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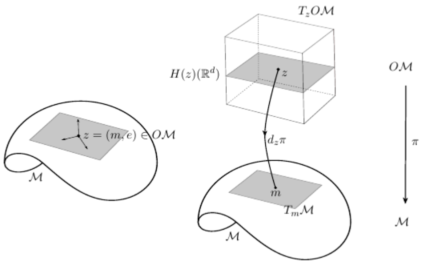
Use **ergodic theorem** and **functional CLT** to conclude. ◁

2. Manifold-valued Kinetic Brownian motion

Let (M, g) be a d-dimensional Riemannian manifold.

- **Cartan development**: a useful way to construct Brownian motion on M. Let $\pi : OM \to M$, stand for the **orthonormal frame bundle** over M; generic point $z = (m, e)$, with e orthonormal basis of $T_m M$. For $z \in OM$, let $H(z) \in L(\mathbb{R}^d, T_z OM)$ stand for the (metric-dependent) horizontal form at z.



$dz_t = H(z_t) \circ dW_t$, in OM, $\qquad m_t := \pi(z_t)$, in M.

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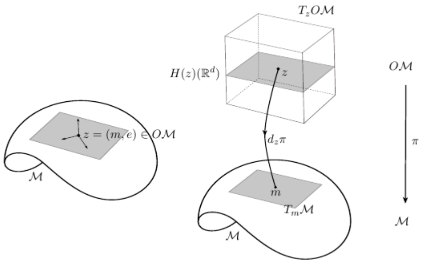

$$dz_t = H(z_t) \circ dW_t, \text{ in OM,} \qquad m_t := \pi(z_t), \text{ in M.}$$

- Definition. **Kinetic Brownian motion m_t^σ in M via Cartan development.** For X_t^σ time rescaled kinetic Brownian motion in $\mathbb{R}^d$, set

$$dz_t^\sigma = H(z_t^\sigma)\, \dot{X}_t^\sigma\, dt, \text{ in OM,} \qquad m_t^\sigma := \pi(z_t^\sigma), \text{ in M.}$$

A parenthesis: Controlled ODEs and rough paths

$$dz_t = V(z_t)dX_t \qquad \big(X_t \in \mathbb{R}^\ell,\ z \in M,\ V(z) \in L(\mathbb{R}^\ell, T_z M)\big)$$

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Robust theory of controlled ODEs for drivers $X \in W^{1,1}$, or $X \in C^\alpha$, for $\alpha > 1/2$, using Young integral formulation: Solution path $z \in C^0$ is a continuous function of control X.

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- A non-linear notion of control = rough paths, elements of a metric space.
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A rough path is an abstract analogue of a tuple of iterated integrals

$$X_t - X_s, \Big(\int_{s\le s_1\le s_2\le t} dX_{s_2}^{i_2}\, dX_{s_1}^{i_1}\Big)_{1\le i_1,i_2\le \ell}, \Big(\int_{s\le s_1\le s_2\le s_3\le t} dX_{s_3}^{i_3}\, dX_{s_2}^{i_2}\, dX_{s_1}^{i_1}\Big)_{1\le i_1,i_2,i_3\le \ell}, \textit{etc.}$$

(These iterated integrals do not make sense when X is α-Hölder with $\alpha \le 1/2$.)

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Definition involves

- a multi-level object indexed by $(0 \le s \le t)$,
- algebraic constraints between its components,
- analytic constraints on the size of its components as functions of (s, t).

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If B is a Brownian motion and $\mathbf{B} = (B, \mathbb{B})$ with

$$\mathbb{B}_{ts} := \int (B_u - B_s) \otimes \circ dB_u.$$

the solution to the rough differential equation

$$dz_t = V(z_t)d\mathbf{B}_t$$

coincides almost surely with the solution of the Stratonovich SDE

$$dz_t = V(z_t)\circ dB_t.$$

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Back to kinetic Brownian motion on a Riemannian manifold M

$$dz^\sigma_t = H(z^\sigma_t)\, dX^\sigma_t, \text{in } OM, \quad m^\sigma_t := \pi(z^\sigma_t), \text{ in } M,$$

with X^σ_t kinetic Brownian motion in $\mathbb{R}^d$.

2. Manifold-valued Kinetic Brownian motion

- Theorem (Bailleul-Angst-Tardif '15) – **Homogenization.** *Assume (M, g) is complete and stochastically complete. Then the process $\left(m_t^\sigma\right)_{0 \le t \le 1}$ converges weakly to a Brownian motion with generator $\frac{4}{d(d-1)}\Delta$, as $\sigma \uparrow \infty$.*

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Idea of proof. Back in $\mathbb{R}^d$ with time rescaled kinetic Brownian motion X_t^σ. Prove that the canonical rough path lift $\mathbf{X}^\sigma$ of $(X_t^\sigma)_{0\leq t\leq 1}$ converges weakly in a rough path sense to the Stratonovich Brownian rough path.

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Use the continuity of the Itô-Lyons solution map for the rough differential equation

$$dz_t^\sigma = H(z_t^\sigma)\, dX_t^\sigma = H(z_t^\sigma)\, d\mathbf{X}_t^\sigma, \quad z_t^\sigma \in OM,$$

to transport weak convergence of $\mathbf{X}^\sigma$ from the rough paths side to the dynamics on OM and M. $\triangleleft$

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- Prove first weak convergence in uniform norm of $\mathbf{X}^\sigma$ to the Stratonovich Brownian rough path, using weak convergence results on stochastic integrals.
- Prove σ-uniform moment bounds on X_{ts}^σ and $\int_s^t X_{us}^\sigma \otimes dX_u^\sigma$, and use Lamperti-type tightness result for random rough paths.

Use the continuity of the Itô-Lyons solution map for the rough differential equation

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3. Anisotropic Kinetic Brownian motion in $\mathbb{R}^d$

Let Σ be a positive-definite symmetric matrix – no loss in assuming $\Sigma = \mathrm{diag}(\alpha_i^2)$.

► Definition. **Anisotropic Kinetic Brownian motion** $(x_t, \dot{x}_t)$ **in** $\mathbb{R}^d$, with anisotropy Σ, is the hypoelliptic diffusion with state space $\mathbb{R}^d \times \mathbb{S}^{d-1}$

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\begin{aligned}
dx_t &= \dot{x}_t \, dt, \\
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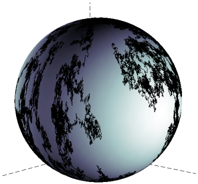
where W is an $\mathbb{R}^d$-valued Brownian motion with covariance Σ, and $P_{\dot{x}} : \mathbb{R}^d \to \langle \dot{x} \rangle^\perp$, the orthogonal projection. (Note $\langle \dot{x} \rangle^\perp = T_{\dot{x}} \mathbb{S}^{d-1}$.)



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- Theorem (Perruchaud '19) – **Homogenization.**
  - *The invariant measure* μ *of the velocity process* $\dot{x}$ *on the sphere is the image by the radial projection on the sphere of the measure on* $\mathbb{R}^d$ *with density* $|x|^{-1}$ *wrt the Gaussian measure with covariance* Σ*.*

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  - *The time-rescaled process* $(x_{\sigma^2 t})_{0\leq t\leq 1}$ *converges weakly as* $\sigma \uparrow \infty$ *to a Euclidean Brownian motion with covariance matrix* $\mathrm{diag}(\gamma_i)$*, with*

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  - We have weak convergence of the associated rough path $\mathbf{X}^\sigma$ to the corresponding Stratonovich Brownian rough path, as $\sigma \uparrow \infty$.

3. Anisotropic Kinetic Brownian motion on $\mathbb{R}^d$

▶ Theorem (Perruchaud '19) – **Homogenization.**

- *The invariant measure* μ *of the velocity process* $\dot{x}$ *on the sphere is the image by the radial projection on the sphere of the measure on* $\mathbb{R}^d$ *with density* $|x|^{-1}$ *wrt the Gaussian measure with covariance* Σ*.*
- *The time-rescaled process* $(x_{\sigma^2 t})_{0 \leq t \leq 1}$ *converges weakly as* $\sigma \uparrow \infty$ *to a Euclidean Brownian motion with covariance matrix* $\mathrm{diag}(\gamma_i)$*, with*

$$\gamma_i := 2 \int_0^\infty \mathbb{E}_\mu\left[\dot{x}_0^i \, \dot{x}_t^i\right] dt, \quad 1 \leq i \leq d.$$

- We have weak convergence of the associated rough path $\mathbf{X}^\sigma$ to the corresponding Stratonovich Brownian rough path, as $\sigma \uparrow \infty$.

Idea of proof. The dynamics of velocity $\dot{x}_t$ is given by the SDE

$$d\dot{x}_t^i = -\frac{\sigma^2}{2}\left(\alpha_i^2 + \sum_{k=1}^d \alpha_k^2 - 2\sum_{\ell=1}^d \alpha_\ell^2 \left|\dot{x}_t^\ell\right|^2\right)\dot{x}_t^i \, dt + \sigma\left(\alpha_i dW_t^i - \dot{x}_t^i \sum_{\ell=1}^d \alpha_\ell \dot{x}_t^\ell dW_t^\ell\right)$$

No clear description of $X_t^\sigma = x_{\sigma^2 t}$, when Σ different from a constant multiple of identity. Give up the analysis of the SDE and **use ergodic properties of** $\dot{x}$.

3. Anisotropic Kinetic Brownian motion on $\mathbb{R}^d$

1. One has for any probability measure λ on $\mathbb{S}^{d-1}$

$$\left\| P_t^* \lambda - \mu \right\|_{\mathrm{TV}} \lesssim e^{-ct},$$

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2. One proves that any limit law turns the canonical process on the rough paths space into a continuous **Lévy process**. We identify its generator using the invariance of the invariant measure μ by the **symmetries**

$$(\theta_1, \dots, \theta_d) \in \mathbb{S}^{d-1} \mapsto (\theta_1, \dots, \theta_{i-1}, -\theta_i, \theta_{i+1}, \dots, \theta_d) \in \mathbb{S}^{d-1}.$$

$\triangleleft$

4. Geometry of the diffeomorphism group

- (M, g) a Riemannian manifold = **domain of the fluid flow**,

  $\mathscr{D} := \{\text{Diffeo of } M\}$ or $H^s(M, M)$: a Fréchet/Hilbert manifold,

  $T_\varphi \mathscr{D} = \{\text{smooth}/H^s \text{ 'vector fields' at } \varphi\} = \{m \in M \to u(m) \in T_{\varphi(m)}M\}$.

(Variant with volume preserving diffeomorphism group and divergence-free vector fields on M.)

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- **Weak Riemannian metric** on $\mathscr{D}$

$$\langle u, v \rangle := \int_M g_{\varphi(m)}\big(u(m), v(m)\big) \,\mathrm{VOL}_g(dm).$$

Induced topology on $\mathscr{D}$ weaker than smooth or H^s topology. There may be no good notion of parallel transport... But Ebin-Marsden (69') prove there is one!
It is a *smooth map*, and its *exponential map* is *well-defined and smooth* in a neighbourhood of the zero section of $T\mathscr{D}$.

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Geodesics (φ_t) on the 'submanifold' of volume preserving diffeomorphisms whose velocity fields $u = \partial_t \varphi_t \circ \varphi_t^{-1}$ are solutions of **Euler's equation for incompressible fluids**

$$\partial_t u + u \nabla u + \nabla p = 0,$$

for a pressure field $p : M \to \mathbb{R}$ ensuring that u remains divergence free.
(V.I. Arnol'd, 66')

4. Geometry of the diffeomorphism group

For the group of volume preserving diffeomorphisms of the 2-dimensional torus $\mathbf{T}^2$:

- Orthonormal basis of the set $\textsc{Lie}(\mathscr{D})$ of null divergence vector fields. For $k \in \mathbb{Z}\backslash\{0\}$

$$A_k = |k|^{-1}\big(k_2 \cos(k \cdot \theta)\partial_1 - k_1 \cos(k \cdot \theta)\partial_2\big),$$
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- Geodesic equation $u := \partial_t\varphi \circ \varphi^{-1}$

$$\partial_t u + \Gamma(u,u) = 0,$$

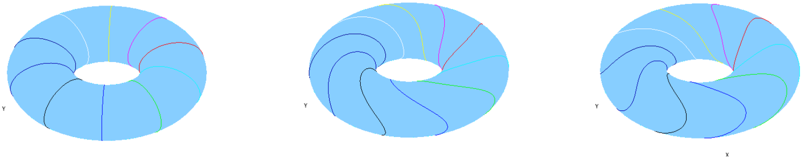
with explicit Christoffel symbols Γ, e.g.

$$\Gamma(A_k, A_\ell) = [k,\ell]\big(\alpha_{k,\ell} B_{k+\ell} + \beta_{k,\ell} B_{k-\ell}\big).$$

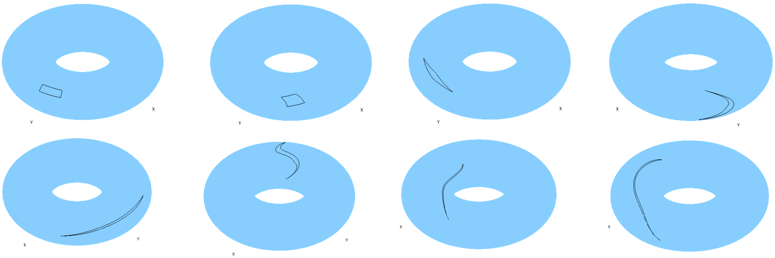
$\Gamma(A_k,\cdot), \Gamma(B_k,\cdot)$ unbounded antisymmetric operators that do not induce nice evolutions on the "orthonormal group" in $\mathrm{Lie}(\mathscr{D})$.

4. Geometry of the diffeomorphism group

- **Time 1 flow with $\sigma = 0$, for different initial momentum in volume preserving diffeomorphism group.**



- **Evolution with time of an area element along geodesic motion in volume preserving diffeomorphism group.**



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We follow Cartan's development strategy, defining first a 'flat' kinetic Brownian motion in the space of vector fields (= the tangent space to identity of the manifold of diffeomorphisms), and then developing it on the manifold of diffeomorphisms.

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1. On $\mathrm{Lie}(\mathscr{D}) \simeq H^s(TM)$. Write $\mathbb{S}$ for unit sphere of $H^s(TM)$,

$$
\begin{aligned}
du_t &= \dot{u}_t \, dt, \\
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2 Follow Ebin-Marsden' strategy, showing one can formulate Cartan's development operation as solving nice ODE on the infinite-dimensional configuration space (= a substitue for the orthonormal frame bundle above $\mathscr{D}$)

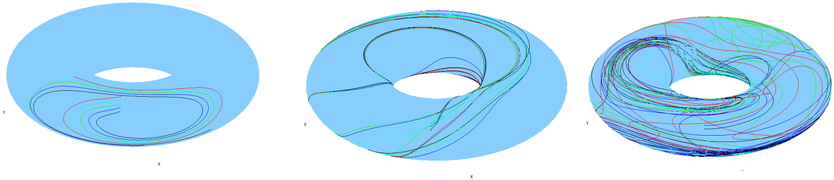
$$TH^s(\mathcal{F}M) \times L(H^s(TM)),$$

   driven by a *smooth* vector field and controlled by u. Set $\varphi_t :=$ projection of dynamics on the diffeomorphism space $\mathscr{D}$.

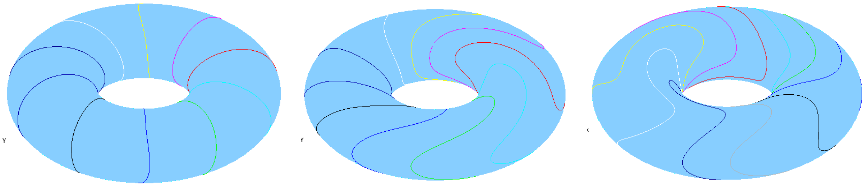
(Variant for volume preserving diffeomorphism group and divergence-free vector fields on M.)

5. Kinetic Brownian motion in the diffeomorphism group

- **Examples of flows with time, for noise parameter $\sigma = 1$.**



- **Time 1 snapshots for increasing noise parameter σ, with same initial momentum.**



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Set $U_t^\sigma := u_{\sigma^2 t} \in \mathrm{LIE}(\mathscr{D})$. Wlog $\Sigma = \mathrm{diag}(\alpha_i^2)$, non-increasing eigenvalues α_i.

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About the proof – Convergence results rely on quantifying the speed of decorrelation of the velocity process $\dot{u}$. Not easy in infinite dimension. Use of conditioning and decorrelation speed to get estimates

$$\sup_{\sigma>0} \mathbb{E}\Big[\big\|X_t^\sigma - X_s^\sigma\big\|^p \vee \big\|\mathbb{X}_{ts}^\sigma\big\|^{p/2}\Big] \lesssim_p |t-s|^{p/2}.$$

5. Kinetic Brownian motion in the diffeomorphism group

Using the above mentioned version of Cartan's development machinery, one can define kinetic Brownian motion in $\mathscr{D}$ in a *small time interval* by solving a rough differential equation driven by the $\mathrm{LIE}(\mathscr{D})$-valued kinetic Brownian motion. (Warning! $\mathscr{D}$ may not be geodesically complete and may have finite diameter!)

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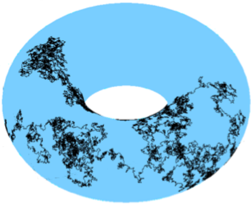
- Theorem (Angst, Bailleul, Perruchaud 19') – **Homogenization in $\mathscr{D}$.** *Kinetic Brownian motion in $\mathscr{D}$ provides an **interpolation** between the dynamics of a(n incompressible) fluid ($\sigma = 0$) and the projection on the diffeomorphism group of a Brownian flow on a larger space ($\sigma = \infty$).*



Thank you!