Fractional Gaussian fields on fractals

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CIRM Workshop: Pathwise Stochastic Analysis and Applications

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where $s \geq 0$, W is a white noise and Δ is the Laplace operator on $\mathbb{R}^n$. The expression (1) has of course to be understood in a distributional sense and means that for every f in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of smooth and rapidly decreasing functions one has

$$\int_{\mathbb{R}^n} (-\Delta)^s f(x) X(dx) = \int_{\mathbb{R}^n} f(x) W(dx).$$

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- When $\frac{n}{4} < s < \frac{n}{4} + \frac{1}{2}$, the Gaussian random measure X admits a density with respect to the Lebesgue measure which is the fractional Brownian motion indexed by $\mathbb{R}^n$. The Hurst parameter H of this fractional Brownian motion is given by $H = 2s - \frac{n}{2}$.

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Regularity theory for fractional Gaussian fields

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Pathwise properties of the fractional Gaussian fields on $\mathbb{R}^n$ are studied using Fourier transform techniques where $(-\Delta)^{-s}$ is seen as the multiplier $\|\lambda\|^{-2s}$ on the Fourier space. In this talk we will show how to define the fractional Gaussian fields and develop their regularity theory on singular spaces like fractals.

Sierpiński gasket

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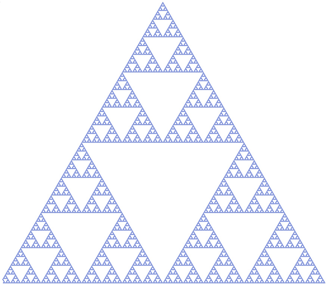

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The explicit values $d_h = \frac{\ln 3}{\ln 2}$ and $d_w = \frac{\ln 5}{\ln 2}$ are known.

Sierpiński gasket

In $\mathbb{R}^2 \simeq \mathbb{C}$, consider the triangle with vertices $q_0 = 0$, $q_1 = 1$ and $q_2 = e^{\frac{i\pi}{3}}$. For $i = 1, 2, 3$, consider the map

$$F_i(z) = \frac{1}{2}(z - q_i) + q_i.$$

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Definition

The Sierpiński gasket is the unique non-empty compact set $K \subset \mathbb{C}$ such that

$$K = \bigcup_{i=1}^{3} F_i(K).$$

Laplacian on the Sierpiński gasket

Denote $V_0 = \{q_0, q_1, q_2\}$. For $f \in C(K)$, one can consider the quadratic forms

$$\begin{aligned}&\mathcal{E}_n(f,f)\\=&\frac{1}{2}\left(\frac{5}{3}\right)^n \sum_{i_1,\cdots,i_n}\sum_{x,y\in V_0}\left(f(F_{i_1}\circ\cdots\circ F_{i_n}(x)) - f(F_{i_1}\circ\cdots\circ F_{i_n}(y))\right)^2\end{aligned}$$

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Define then

$$\mathcal{F} = \left\{ f \in C(K), \lim_{n\to\infty} \mathcal{E}_n(f,f) < +\infty \right\}$$

and for $f \in \mathcal{F}$,

$$\mathcal{E}(f,f) = \lim_{n\to\infty} \mathcal{E}_n(f,f).$$

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$$\mathcal{E}(f, g) = \frac{5}{3} \sum_{i=1,2,3} \mathcal{E}(f \circ F_i, g \circ F_i).$$

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The generator Δ of this Dirichlet form is called the Laplacian of the gasket.

White noise on the Sierpiński gasket

We consider on the measurable space $(K, \mathcal{K}, \mu)$, where $\mathcal{K}$ is the Borel σ-field on K, a real-valued Gaussian random measure $W : \mathcal{K} \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ with intensity μ. In other words, W is such that

- a.s. W is a measure on $(K, \mathcal{K})$
- for any $A \in \mathcal{K}$ such that $\mu(A) < \infty$, $W(A)$ is a real-valued Gaussian variable with mean zero and variance $\mathbb{E}\left(W(A)^2\right) = \mu(A)$
- for any sequence $(A_n)_{n\in\mathbb{N}} \in \mathcal{K}^{\mathbb{N}}$ of pairwise disjoint measurable sets, the random variables $W(A_n)$, $n \in \mathbb{N}$, are independent.

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Lemma

If $s > \frac{d_h}{2d_w}$, the Gaussian random measure $(-\Delta)^{-s} W$ has a density X given by

$$X(x) = \int_K G_s(x,z)\, W(dz),\ x \in K,$$

where G_s is the integral kernel of the operator $(-\Delta)^{-s}$.

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The keypoint is that for $s > \frac{d_h}{2d_w}$, there exists an L^2 kernel $G_s(x,y)$ such that

$$(-\Delta)^{-s} f = \int_K G_s(\cdot, y) f(y) d\mu(y).$$

Fractional Riesz kernels on the Sierpiński gasket

The fractional Riesz kernel G_s is given by the formula

$$G_s(x,y) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1}(p_t(x,y) - 1)dt.$$

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$$p_t(x,y) \simeq c_1 t^{-d_h/d_w} \exp\left(-c_2 \left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right)$$

Hölder regularity of the FGFs

To study the existence of a Hölder continuous version for the density X one can appeal to the entropy method (Adler-Taylor) for Gaussian fields which is a technique available in Ahlfors regular metric spaces.

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Theorem

For $s \in \left(\frac{d_h}{2d_w}, 1 - \frac{d_h}{2d_w}\right)$ *and* $f \in L^2(K,\mu)$

$$|(-\Delta)^{-s} f(x) - (-\Delta)^{-s} f(y)| \leq C d(x,y)^{s d_w - \frac{d_h}{2}} \|f\|_{L^2(K,\mu)}.$$

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$$|(-\Delta)^{-s} f(x) - (-\Delta)^{-s} f(y)| \le C d(x,y)^{s d_w - \frac{d_h}{2}} \|f\|_{L^2(K,\mu)}.$$

Therefore, in the range $s \in \left(\frac{d_h}{2d_w}, 1 - \frac{d_h}{2d_w}\right)$, the operator $(-\Delta)^{-s}$ maps $L^2(K,\mu)$ into the space of bounded and $\left(s d_w - \frac{d_h}{2}\right)$-Hölder continuous functions.

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The proof partially relies upon the theory of Besov spaces on Dirichlet spaces that was recently developed by Alonso-Ruiz, B., Chen, Rogers, Shanmugalingam and Teplyaev.

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The proof partially relies upon the theory of Besov spaces on Dirichlet spaces that was recently developed by Alonso-Ruiz, B., Chen, Rogers, Shanmugalingam and Teplyaev. In particular, from that theory it is known that for the Sierpiński gasket there exists a constant $C > 0$ such that for every $g \in L^\infty(K, \mu)$, $t > 0$ and $x, y \in K$,

$$|P_t g(x) - P_t g(y)| \leq C \frac{d(x, y)^{d_w - d_h}}{t^{1 - \frac{d_h}{d_w}}} \|g\|_{L^\infty(K,\mu)}.$$

Hölder regularity of the FGFs

Idea of the proof: One has

$$\begin{aligned}
&\left|(-\Delta)^{-s}f(x) - (-\Delta)^{-s}f(y)\right| \\
=&C\left|\int_0^{+\infty} t^{s-1}(P_tf(x) - P_tf(y))dt\right| \\
\leq&C\int_0^{+\infty} t^{s-1}\left|P_tf(x) - P_tf(y)\right| dt
\end{aligned}$$

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Idea of the proof: One has

$$\begin{aligned}
&\left|(-\Delta)^{-s} f(x) - (-\Delta)^{-s} f(y)\right| \\
=&C \left| \int_0^{+\infty} t^{s-1} (P_t f(x) - P_t f(y)) dt \right| \\
\le&C \int_0^{+\infty} t^{s-1} \left| P_t f(x) - P_t f(y) \right| dt
\end{aligned}$$

We then decompose the integral

$$\int_0^{+\infty} = \int_0^{\delta} + \int_{\delta}^{+\infty}$$

with $\delta \simeq d(x,y)^{d_w}$.

Hölder regularity of the FGFs

The small time integral $\int_0^\delta$ is controlled using ultracontractivity of the semigroup P_t and the large time integral $\int_\delta^{+\infty}$ is controlled using interpolation theory and the Hölder regularization estimate for P_t.

Hölder regularity of the FGFs

Coming back to the regularity problem for X, applying the entropy method we obtain

Theorem

There exists a modification X^* *of* $(-\Delta)^{-s}W$ *such that*

$$\lim_{\delta\to 0} \sup_{\substack{d(x,y)\leq\delta \\ x,y\in K}} \frac{|X^*(x)-X^*(y)|}{d(x,y)^H\sqrt{|\ln d(x,y)|}} < \infty,$$

where $H = sd_w - \frac{d_h}{2}$

Maximal Hölder regularity of the FGFs

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For the gasket, the optimal Hölder regularity exponent of the FGFs is $H = d_w - d_h$. For other fractals, the method we developed also works in the range $\frac{d_h}{2d_w} < s \leq 1 - \frac{d_h}{2d_w}$, however the optimal Hölder regularity exponent of the FGFs is unknown and conjectured to be

$$H = d_w - d_h + d_{tH} - 1$$

where d_{tH} is the topological Hausdorff dimension of the carpet.