The non-commutative signature of a path

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Joint work with N. Gilliers (Universität Greifswald)

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NC probability theory

This talk wants to arouse again interest for rough paths principles applied to non-commutative probability theory.

We suppose given a unital C^* algebra $(\mathcal{A}, \mathbf{1}, \cdot, *, \|\cdot\|)$ and a trace $\varphi\colon \mathcal{A} \to \mathbb{C}$, $\varphi(\mathbf{1}) = 1$, $\varphi(ab) = \varphi(ba)$, $\varphi(aa^*) \geq 0$.

Basic examples:

- $\mathcal{A} = L^\infty(\Omega, \mathbb{P})$ and $\varphi = \mathbb{E}$ (the only commutative example)
- $\mathcal{A} = M_{N\times N}(L^\infty(\Omega, \mathbb{P}))$ and $\varphi(A) = \frac{1}{N}\mathbb{E}(\mathrm{Tr}\, A)$
- $\mathcal{A}$ is an infinite-dimensional VN Algebra

NC random variables X are self-adjoint elements of $\mathcal{A}$
NC stochastic processes are self-adjoint paths $X\colon [0,1] \to \mathcal{A}$.
In this talk, we will focus only on the space $\mathcal{A}$.

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NC stochastic calculus

During the last 30 years several theories of NC stochastic calculus theories where introduced to define a NC stochastic integral

$$\int_0^t A_s \cdot dX_s \cdot B_s$$

Some fundamental examples:

- $\mathcal{A}$ is the Fock-space of a Hilbert space and X is the annihilation operator (quantum stochastic calculus) [Parthasarathy '84]
- $\mathcal{A}$ is a Boolean Fock space and X is the preservation operator (boolean stochastic calculus) [Ghorbal-Schürmann '02]
- $\mathcal{A}$ is a VN Algebra and X is a free Brownian motion (free stochastic calculus) [Biane-Speicher '98]

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NC stochastic differential equations

A class of NC SDE to study with rough paths is

$$\mathrm{d}Y_t = f(Y_t) \cdot \mathrm{d}X_t \cdot g(Y_t)\,, \quad Y_0 \in \mathcal{A}$$

with $f, g\colon \mathcal{A} \to \mathcal{A}$ smooth in terms of functional calculus.

- These equations might arise as limit in law for random matrices models in large dimension
- The standard theory to solve them studies equations with additive noise, relying strongly on standard Itô theory, is still lacking a strong solution theory

Is rough path theory capable of studying these equations? Before that, does X lift to an explicit rough path?

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NC rough paths (standard theory)

The infinite-dimensional setting makes the theory more complicate [Ledoux-Lyons-Qian '02]. Due to the presence of a product in the equations, it is mandatory to consider a projective tensor product.

- In case of Free and q-Brownian Motion, using some sharp BDG inequalities, there exist Levy areas in the spatial tensor product [Capitaine, Donati-Martin '01][Victoir '04]
- Using the abstract results from [Lyons-Victoir '07] it is possible to construct a geometric rough path which could not coincide with general Wong-Zakai approximation schemes
- In the case $\gamma \in (1/3, 1/2)$ similar problems were recently studied in [Grong-Nilssen-Schmeding '20] using sub-Riemannian geometry in infinite dimension

The Deya-Schott's approach

In [Deya-Schott '13] the authors introduced a new object to study the solution of

$$dY_t = f(Y_t) \cdot dX_t \cdot g(Y_t) \quad Y_0 \in \mathcal{A}$$

We fix $A, B \in \mathcal{A}$ and we consider the linearised equation

$$dY_t = (A \cdot Y_t) \cdot dX_t \cdot (Y_t \cdot B) \quad Y_0 = \mathbf{1}$$

Writing down the Picard iterations

$$Y_t = \mathbf{1} + A \cdot \delta_{0t} X \cdot B + \int_{\Delta^2_{0t}} A^2 \cdot \mathrm{d}X_{t_1} \cdot B \cdot \mathrm{d}X_{t_2} \cdot B + \int_{\Delta^2_{0t}} A \cdot \mathrm{d}X_{t_2} \cdot A \cdot \mathrm{d}X_{t_1} \cdot B^2 + (\cdots)$$

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The Deya-Schott's approach

If we arrest the expansion at level 2 and we look at two instants $s < t$ it is possible to rewrite the expansion using the formal object

$$(s,t) \to \left[(A,B) \to \int_{\Delta^2_{st}} A \cdot \mathrm{d}X_{t_1} \cdot B \cdot \mathrm{d}X_{t_2}\right] \in L(\mathcal{A} \otimes \mathcal{A}, \mathcal{A})$$

This object is at the basis of the product Lévy area above X [Deya-Schott '13]. The precise definition requires the projective tensor product and some measurability conditions in the inputs.

The product Levy area is weaker than the Lévy area and it can be constructed explicitly from dyadic partitions.

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Higher order expansion

What are the next terms in the expansion?

$$Y_t = \mathbf{1} + A(\delta_{0t}X)B + \int_{\Delta^2_{0t}} A^2 \mathrm{d}X_{t_1} B \mathrm{d}X_{t_2} B + \int_{\Delta^2_{0t}} A\mathrm{d}X_{t_2} A\mathrm{d}X_{t_1} B^2$$

$$+ \int_{\Delta^3_{0t}} A^3 \mathrm{d}X_{t_1} B\mathrm{d}X_{t_2} B\mathrm{d}X_{t_3} B + \int_{\Delta^3_{0t}} A^2 \mathrm{d}X_{t_2} A\mathrm{d}X_{t_1} B^2 \mathrm{d}X_{t_3} B$$

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The solution is described using permutations and contractions of the third order integral $\int_{\Delta^3_{0t}} \mathrm{d}X_{t_1} \otimes \mathrm{d}X_{t_2} \otimes \mathrm{d}X_{t_3}$.

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Full contractions

Definition

Let $X\colon [0,1] \to \mathcal{A}$ be a smooth path and $\sigma \in \mathcal{S}_n$. We define the full contraction along σ as $\mathbf{X}^\sigma\colon [0,1]^2 \to L(\mathcal{A}^{\otimes(n+1)}, \mathcal{A})$

$$\mathbf{X}^\sigma_{st}(A_0, \dots, A_n) = \int_{\Delta^n_{st}} A_0 \mathrm{d}X_{t_{\sigma(1)}} A_1 \cdots A_{n-1} \mathrm{d}X_{t_{\sigma(n)}} A_n$$

Instead of elements living in $T(\mathcal{A}) = \bigoplus_{n=1}^\infty \mathcal{A}^{\otimes n}$, the full contractions live in the endomorphism space

$$\mathrm{End}(\mathcal{A}) = \bigoplus_{n=1}^\infty L\left(\mathbb{C}[\mathcal{S}_n], L(\mathcal{A}^{\otimes(n+1)}, \mathcal{A})\right)$$

Moreover there exists a linear map $\mathrm{Op}\colon T(\mathcal{A}) \to \mathrm{End}(\mathcal{A})$ transforming signatures in full contractions.

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Moreover there exists a linear map $\mathrm{Op}\colon T(\mathcal{A}) \to \mathrm{End}(\mathcal{A})$ transforming signatures in full contractions.

Is it possible to write down a Chen relation on full contractions?

$$\mathbf{X}^3_{st} = \mathbf{X}^{(132)}_{st}(A_0, A_1, A_2, A_3) = \int_{\Delta^3_{st}} A_0 \mathrm{d}X_{t_1} A_1 \mathrm{d}X_{t_3} A_2 \mathrm{d}X_{t_2} A_3$$

$$\mathbf{X}^3_{st} - \mathbf{X}^3_{su} - \mathbf{X}^3_{ut} = \int_{(t_1,t_2)\in\Delta^2_{su}} \int_{t_3\in\Delta^1_{ut}} A_0 \mathrm{d}X_{t_1} A_1 \mathrm{d}X_{t_3} A_2 \mathrm{d}X_{t_2} A_3 + \int_{t_1\in\Delta^1_{su}} \int_{(t_2,t_3)\in\Delta^2_{ut}} A_0 \mathrm{d}X_{t_1} A_1 \mathrm{d}X_{t_3} A_2 \mathrm{d}X_{t_2} A_3$$

In the non-commutative case we cannot express this term using products of full contractions at lower order.

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The need of partial contractions

By enlarging the family of full contractions and studying a different operation, we can write down a new identity.

Let $\mathbf{X}^{(1)}_{st} \in L(\mathcal{A}^2, \mathcal{A})$ and $\mathbf{Y}_{st} \in L(\mathcal{A}^4, \mathcal{A}^2)$ be the maps

$$\mathbf{X}^{(1)}_{st}(A_0, A_1) = A_0(\delta_{st}X)A_1$$

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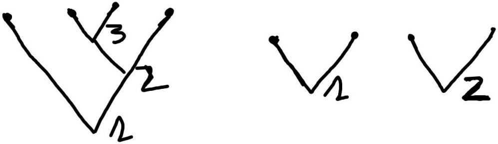
Levelled trees and forests

Definition

A levelled tree/forest is a planar binary rooted tree/forest endowed with an decoration on the interior nodes which preserves the intrinsic partial order union the root tree and the empty forest.



We denote by $\mathcal{T}_n$ the set of levelled trees with n leaves and $\mathcal{F}^n_m$ the set of levelled forests with n leaves and m trees. These objects describe efficiently full and partial contractions.

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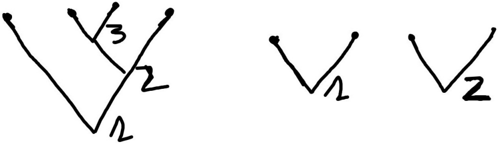

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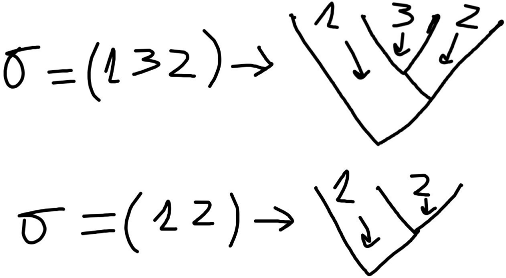
Permutations and trees

Theorem (Loday-Ronco '98)

The set $\mathcal{T}_{n+1}$ is in bijection with $\mathcal{S}_n$.

We write $\sigma = (\sigma(1)\cdots\sigma(n))$ to obtain a decoration



We encode full contractions using a meaningful structure.

Partial contractions and forests

Definition

Let $X\colon [0,1] \to \mathcal{A}$ be a smooth path and $f \in \mathcal{F}_m^{n+1}$ such that $f = f_1 \cdots f_m$ where $f_k \in \mathcal{S}_{n_k}$. We define the partial contraction along f as $\mathbf{X}^f\colon [0,1]^2 \to L(\mathcal{A}^{\otimes(n+1)}, \mathcal{A}^{\otimes m})$

$$\mathbf{X}^f_{st}(B_0, \dots, B_n) = \int_{\Delta_{st}^{n+1-m}} \mathrm{d}X^{f_1}_{t_{f_1}}(B_0, \cdots) \otimes \cdots \otimes \mathrm{d}X^{f_m}_{t_{f_m}}(\cdots, B_{n_m})$$

$$\mathrm{d}X^{f_k}_{t_{f_k}}(C_0, \cdots, C_{n_k}) = C_0 \mathrm{d}X_{t_{f_k}(1)} C_1 \cdots \mathrm{d}X_{t_{f_k}(n_k)} C_k$$

Partial contractions live in the bigraded structure

$$\mathrm{Mult}(\mathcal{A}) = \bigoplus_{n=1}^{\infty} \bigoplus_{m=1}^{n+1} L\left(\mathbb{C}[\mathcal{F}_m^{n+1}], L(\mathcal{A}^{\otimes(n+1)}, \mathcal{A}^{\otimes m})\right)$$

and there exists an explicit linear map m-Op: $T(\mathcal{A}) \to \mathrm{Mult}(\mathcal{A})$.

Partial contractions and forests

Definition

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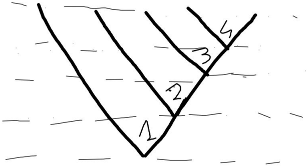
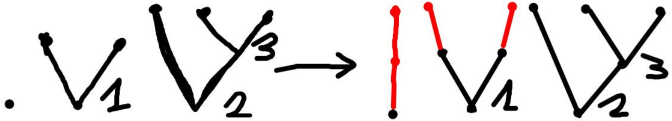
Operations on forests

Every levelled tree can be cut all along its vertical generations



Looking at levelled forests f, we add extra nodes and edges to keep the number of generations $G(f)$ constant.



Operations on forests

By cutting each forest f along its i-th generation we obtain two subforests f_i^+ f_i^- and a coproduct operation

$$\Delta f = \sum_{i=0}^{G(f)} f_i^- \otimes f_i^+$$

The proper spaces to describe Δ are forest $\mathbb{C}[\mathcal{F}]$ and couples of forests with some compatibility conditions on the vertical generations $\mathbb{C}[\mathcal{F}] \obar \mathbb{C}[\mathcal{F}]$. $\obar$ is called vertical tensor product.

Theorem (B. Gilliers '21)

There exists a product $\mu \colon \mathbb{C}[\mathcal{F}] \obar \mathbb{C}[\mathcal{F}] \to \mathbb{C}[\mathcal{F}]$ such that $(\mathbb{C}[\mathcal{F}], \mu, \Delta)$ is a Hopf algebra structure with respect to $\obar$.

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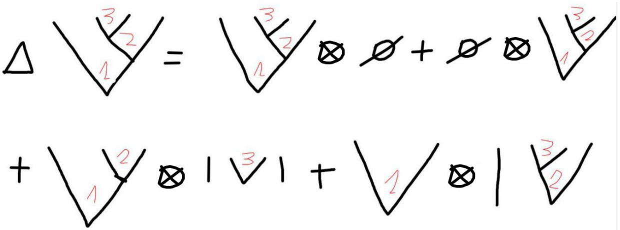
Example



$$\mathbf{X}^3_{st} = \mathbf{X}^3_{su} + \mathbf{X}^3_{ut} + \int_{(t_1,t_2)\in\Delta^2_{su}} \int_{t_3\in\Delta^1_{ut}} \cdots + \int_{t_1\in\Delta^1_{su}} \int_{(t_2,t_3)\in\Delta^2_{ut}} \cdots$$

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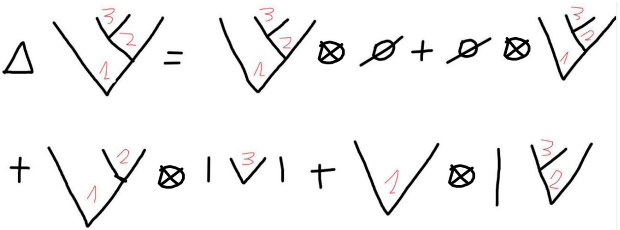

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The NC Signature

Theorem (B., Gilliers '21)

Let $X\colon [0,1] \to \mathcal{A}$ be a smooth path with signature $\mathbb{X}\colon [0,1]^2 \to T(\mathcal{A})$. Defining $\mathbf{X} = m\text{-}Op(\mathbb{X})$, we have a function $\mathbf{X}\colon [0,1]^2 \to \mathrm{Mult}(\mathcal{A})$ satisfying the following properties:

- *For any levelled $f \in \mathcal{F}$ and $s, u, t \in [0,1]$*

$$\mathbf{X}^f_{st} = (\mathbf{X}_{ut} \circ \mathbf{X}_{su})\Delta f \quad \textit{(NC Chen relations)}$$

- *For any levelled forest f, h such that $\mathbf{X}^f_{st} \circ \mathbf{X}^h_{st}$ is well-defined*

$$\mathbf{X}_{st}(\mu(f,h)) = \mathbf{X}^f_{st} \circ \mathbf{X}^h_{st} \quad \textit{(NC shuffle relations)}$$

We call $\mathbf{X}$ the NC signature of X.

Remarks and perspectives

- Similarly to standard signatures, there exists a Lie group G such that $\mathbf{X}\colon [0,1]^2 \to G$ are group increments.

- The value of $\mathbf{X}$ is uniquely determined by full-contractions and an extra class of operations called faces-contractions

- Using this definition it is possible to design a notion of NC rough paths and NC controlled rough paths. How do these definitions behave when there is a trace φ

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Thanks