Scaling limit results for additive functionals of mixed fractional Brownian motions

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In joint work with Khoa Lê (TU Berlin)

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A family of occupation densities for $(X_t)_{t\geq 0}$ is defined (here) as a process $(L_t^a)_{a\in\mathbb{R}^d, t\geq 0}$ such that, almost-surely, we have the *occupation times formula*

$$\int_0^t f(X_s)\, ds = \int_{\mathbb{R}^d} f(a)\, L_t^a\, da,$$

valid for all $f : \mathbb{R}^d \to \mathbb{R}_+$ Borel measurable.

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There exist various approaches to the construction of occupation densities, that apply in different contexts. Standard constructions apply when X is:

- a real-valued semi-martingale,
- a Markov process,
- a Gaussian process with an appropriate covariance.

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In this talk we will address the second application.

Theorem (Darling-Kac Theorem)

Let (B_t) be a standard Brownian motion in $\mathbb{R}$, and $(L_t^a(B))_{a\in\mathbb{R},t\geq 0}$ the associated family of occupation densities. Let $f \in L^1(\mathbb{R})$ and $t \geq 0$ fixed. Then, as $\lambda \to \infty$, $\frac{1}{\sqrt{\lambda}}\int_0^{\lambda t} f(B_s)\,ds$ converges in distribution to $(\int_{\mathbb{R}} f)\,L_t^0(B)$.

Proof: Performing the change of variable $s = \lambda u$, and by the scaling property of Brownian motion, we have

$$\frac{1}{\sqrt{\lambda}}\int_0^{\lambda t} f(B_s)\,ds = \sqrt{\lambda}\int_0^t f(B_{\lambda u})\,du \overset{(d)}{=} \sqrt{\lambda}\int_0^t f(\sqrt{\lambda}B_u)\,du.$$

By the occupation times formula, and performing the change of variable $b = \sqrt{\lambda}a$, the latter equals

$$\sqrt{\lambda}\int_{-\infty}^{+\infty} f(\sqrt{\lambda}a)\,L_t^a(B)da = \int_{-\infty}^{+\infty} f(b)\,L_t^{b/\sqrt{\lambda}}(B)\,db.$$

As $\lambda \to +\infty$, by dominated convergence, the latter integral converges to $\left(\int_{-\infty}^{+\infty} f(b)\,db\right) L_t^0(B)$, whence the claim.

Let now $(B_t^H)_{t\geq 0}$ be a d-dimensional fractional Brownian motion (fBM) of Hurst parameter $H \in (0,1)$. That is, $B_t^H = (B_t^{H,1}, \dots, B_t^{H,d})$ is a centered Gaussian process with covariance matrix

$$\mathbb{E}[B_s^{H,i}\, B_t^{H,j}] = \delta_{i,j} \frac{|t|^{2H} + |s|^{2H} - |t-s|^{2H}}{2}, \qquad 1 \leq i,j \leq d.$$

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$(B_t^H)_{t\geq 0}$ is scale-invariant:

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Assume $Hd < 1$. Then one can prove (c.f. Geman-Horowitz, 1980) that $(B^H_t)_{t\geq 0}$ admits jointly continuous occupation densities $(L^a_t(B^H))_{a\in\mathbb{R}^d, t\geq 0}$.

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One can further prove limit theorems for additive functionals of $(B^H_t)_{t\geq 0}$.

Theorem (c.f. Hu-Nualart-Xu, 2014)

Assume that $Hd < 1$. Let (B_t^H) be a d-dimensional fBM of Hurst parameter H, and let $(L_t^a(B^H))_{a\in\mathbb{R}^d, t\geq 0}$ be the associated family of occupation densities. Let $f \in L^1(\mathbb{R}^d)$ and $t \geq 0$ fixed. Then, as $\lambda \to \infty$, $\lambda^{Hd-1}\int_0^{\lambda t} f(B_s^H)\,ds$ converges in distribution to $\left(\int_{\mathbb{R}^d} f\right) L_t^0(B^H)$.

Proof: Performing the change of variable $s = \lambda u$, and by the scaling property of the fBM, we have

$$\lambda^{Hd-1}\int_0^{\lambda t} f(B_s^H)\,ds = \lambda^{Hd}\int_0^t f(B_{\lambda u}^H)\,du \overset{(d)}{=} \lambda^{Hd}\int_0^t f(\lambda^H B_u^H)\,du.$$

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As $\lambda \to +\infty$, by dominated convergence, the latter integral converges to $\left(\int_{\mathbb{R}^d} f(b)\,db\right)\, L_t^0(B^H)$, whence the claim.

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2. scale invariance of the underlying process

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1. existence of a jointly continuous family of occupation densities
2. scale invariance of the underlying process

What can we say for a non-Markovian process that is not scale-invariant?

A non-scale-invariant process

We consider the process $X_t := B_t + \alpha B_t^H$, where $\alpha \in \mathbb{R} \setminus \{0\}$ and:

- B is a d-dimensional BM,
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X is called a *mixed fractional Brownian motion*. It is not a semi-martingale for $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{3}{4}\right]$ (Cheridito, 2001). Moreover, it is not scale-invariant for $H \neq 1/2$.

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Question: Can one prove limit theorems for additive functionals of such a process?

Let $X_t := B_t + \alpha B_t^H$ be a mixed fBM.

Theorem (E, Lê, 2021+)

X admits a jointly continuous family $(L_t^a)_{a\in\mathbb{R}^d, t\geq 0}$ of occupation densities whenever $(H \wedge \frac{1}{2})d < 1$. Moreover, for all $f \in L^1(\mathbb{R}^d)$ and any fixed $t \geq 0$, the following limits hold:

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1. *when $H < \frac{1}{2}$ and $d = 1$: $\lambda^{-\frac{1}{2}} \int_0^{\lambda t} f(X_r)dr$ converges as $\lambda \to \infty$ in distribution to $\int_{\mathbb{R}} f(y)dy\, L_t^0(B)$,*

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In the sequel we shall sketch the proof of existence of the occupation densities when $(H \wedge \frac{1}{2})d < 1$ and of the first limit.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space, $m \geq 2$ and $T > 0$ fixed. We write L^m for $L^m(\Omega, \mathcal{F}, \mathbb{P})$ and, for all $s \geq 0$, we write $\mathbb{E}^s$ for $\mathbb{E}[\cdot|\mathcal{F}_s]$.

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For $0 \leq s \leq t \leq T$, we assume given $A_{s,t} \in L^m$. For $0 \leq s \leq u \leq t \leq T$, we further set

$$\delta A_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t}.$$

Theorem (Stochastic Sewing Lemma, K. Lê)

Assume that there exist $\Gamma_1, \Gamma_2 \geq 0$ *and* $\epsilon_1, \epsilon_2 > 0$, *such that*

$$\|\mathbb{E}^s[\delta A_{s,u,t}]\|_{L^m} \leq \Gamma_1 \left|t-s\right|^{1+\epsilon_1} \tag{1}$$

and

$$\|\delta A_{s,u,t} - \mathbb{E}^s[\delta A_{s,u,t}]\|_{L^m} \leq \Gamma_2 \left|t-s\right|^{\frac{1}{2}+\epsilon_2}. \tag{2}$$

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Then there exist a constant $C = C(m, \epsilon_1, \epsilon_2)$ *and an adapted process* $(\mathcal{A}_t)_{0 \leq t \leq T}$ *in* L^m *such that* $\mathcal{A}_0 = 0$ *and satisfying*

$$\|\mathcal{A}_t - \mathcal{A}_s - A_{s,t}\|_{L^m} \leq C\Gamma_1 \, |t-s|^{1+\epsilon_1} + C\Gamma_2 \, |t-s|^{\frac{1}{2}+\epsilon_2}$$

and

$$\|\mathbb{E}^s[\mathcal{A}_t - \mathcal{A}_s - A_{s,t}]\|_{L^m} \leq C\Gamma_1 \, |t-s|^{1+\epsilon_1}.$$

Such a process $\mathcal{A}$ *is unique up to modification.*

Remark: If there exist $\epsilon > 0$ and $\alpha \in [0,1)$ such that $\|A_{s,t}\|_{L^m} \leq \Gamma|t-s|^{\frac{1}{2}+\epsilon}$, and $\mathbb{E}^s[\delta A_{s,u,t}] = 0$, then the assumptions of the SSL are fulfilled, and we have

$$\|\mathcal{A}_t - \mathcal{A}_s - A_{s,t}\|_{L^m} \lesssim \Gamma|t-s|^{\frac{1}{2}+\epsilon},$$

whence, by the triangle inequality

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Corollary (Stability with respect to the germ)

If $A_{s,t}$ and $\tilde{A}_{s,t}$ are two germs such that $\|A_{s,t} - \tilde{A}_{s,t}\|_{L^m} \leq \Gamma |t-s|^{\frac{1}{2}+\epsilon}$, and satisfying $\mathbb{E}^s[\delta A_{s,u,t}] = \mathbb{E}^s[\delta \tilde{A}_{s,u,t}] = 0$, then for all $t \geq 0$,

$$\|\mathcal{A}_t - \tilde{\mathcal{A}}_t\|_{L^m} \lesssim \Gamma\, t^{\frac{1}{2}+\epsilon}.$$

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Proof: apply the above Remark to $A - \tilde{A}$.

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Assume that, for some $\alpha_1 \in [0,1)$ and $\alpha_2 \in [0,1/2)$, we have

$$\|\mathbb{E}^s[\delta A_{s,u,t}]\|_{L^m} \leq \Gamma_1\, u^{-\alpha_1}\, |t-s|^{1+\epsilon_1} \tag{3}$$

and

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Then the conclusions of the SSL holds, but the estimates are replaced with

$$\begin{aligned} \|\mathcal{A}_t - \mathcal{A}_s - A_{s,t}\|_{L^m} \leq C\Gamma_1 \left(\int_s^t u^{-\alpha_1} \, du \right) |t - s|^{\epsilon_1} \\ + C\Gamma_2 \left(\int_s^t u^{-2\alpha_2} \, du \right)^{1/2} |t - s|^{\epsilon_2}, \end{aligned}$$

and

$$\|\mathbb{E}^s[\mathcal{A}_t - \mathcal{A}_s - A_{s,t}]\|_{L^m} \leq C\Gamma_1 \left(\int_s^t u^{-\alpha_1} \, du \right) |t - s|^{\epsilon_1}.$$

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Proposition (Existence of occupation densities)

Let $d \geq 1$ *and* $H \in (0,1)$ *such that* $\left(H \wedge \frac{1}{2}\right) d < 1$*, and let* $\alpha \in \mathbb{R} \setminus \{0\}$*. Then* $X_t := B_t + \alpha B_t^H$ *admits a family of occupation densities* $(L_t^a)_{a \in \mathbb{R}^d, t \geq 0}$*. Moreover, for all* $a \in \mathbb{R}^d$*,* L_t^a *is in* L^m *for all* $m \geq 2$ *such that* $(H \wedge \frac{1}{2})d < \frac{m}{2(m-1)}$*.*

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Proposition (Scaling limit for $d = 1, H < 1/2$)

Assume that $H < \frac{1}{2}$ and $d = 1$. Then, for all $f \in L^1(\mathbb{R})$ and $t \geq 0$ fixed, $\lambda^{-\frac{1}{2}} \int_0^{\lambda t} f(X_r) dr$ converges as $\lambda \to \infty$ in distribution to $\int_{\mathbb{R}} f(y) dy \, L_t^0(B)$.

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We construct the fBM $(B_t^H)_{t\geq 0}$ using the Mandelbrot-van Ness representation:

$$B_t^H = c_H \int_{-\infty}^{t} \left[(t-r)_+^{H-1/2} - (-r)_+^{H-1/2} \right] dW_r,$$

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Finally we set $X_t = B_t + \alpha B_t^H$. Note that $(X_t)_{t\geq 0}$ is adapted w.r.t. $(\mathcal{F}_t)_{t\geq 0}$.

Proof idea for the existence statement Let $a \in \mathbb{R}^d$. We would like to construct L_t^a using the SSL. Formally, $L_t^a = \int_0^t \delta_a(X_s)\, ds$. What would be a good germ?

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But let us consider instead $\tilde{A}_{s,t} := \int_s^t \mathbb{E}^s(g(X_r))\, dr$. Then $\tilde{A}_{s,t}$ satisfies the assumptions of the SSL. Indeed, $\delta \tilde{A}_{s,u,t} = \int_u^t (\mathbb{E}^s[g(X_r)] - \mathbb{E}^u[g(X_r)])\, dr$, hence $\mathbb{E}^s[\delta \tilde{A}_{s,u,t}] = 0$, and

$$\|\delta \tilde{A}_{s,u,t}\|_{L^m} \leq \int_u^t \|\mathbb{E}^s[g(X_r)] - \mathbb{E}^u[g(X_r)]\|_{L^m}\, dr \leq 2\|g\|_\infty |t-s|.$$

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Moreover, the germ $\tilde{A}_{s,t}$ also generates the process $\mathcal{A}_t$. Indeed $\mathbb{E}^s[\mathcal{A}_t - \mathcal{A}_s - \tilde{A}_{s,t}] = 0$ and

$$\|\mathcal{A}_t - \mathcal{A}_s - \tilde{A}_{s,t}\|_{L^m} = \left\| \int_s^t (g(X_r) - \mathbb{E}^s[g(X_r)])\, dr \right\|_{L^m} \leq 2\|g\|_\infty |t-s|.$$

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Fact: For all $s \le t$, we can write $X_t = \mathbb{E}^s(X_t) \overset{\perp\!\!\!\perp}{+} Z_{s,t}$, with

- $\mathbb{E}^s(X_t) \sim \mathcal{N}(0, \kappa(s,t) I_d)$, where $\kappa(s,t) \gtrsim s^{2\left(H \wedge \frac{1}{2}\right)}$,
- $Z_{s,t} \sim \mathcal{N}(0, \rho(s,t) I_d)$, where

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Therefore, with $p_t(x) := \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{\|x\|^2}{2t}\right)$ the heat kernel in $\mathbb{R}^d$, we have

$$\mathbb{E}^s(\delta_a(X_u)) = \mathbb{E}^s(\delta_a(\mathbb{E}^s(X_u) + Z_{s,u})) = p_{\rho(s,u)}(a - \mathbb{E}^s(X_u)).$$

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In particular $\mathbb{E}^s[\delta A_{s,u,t}] = 0$. To bound $\|\delta A_{s,u,t}\|_{L^m}$, it suffices to bound $\|p_{\rho(v,r)}(a - \mathbb{E}^v(X_r))\|_{L^m}$ for any $v < r$. We use the following:

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Let $\kappa, \rho > 0$, $X \sim \mathcal{N}(0, \kappa\, I_d)$, and $a \in \mathbb{R}^d$. Then, for all $m \geq 2$,

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Proof: Assume for simplicity $m < \infty$. Then

$$\mathbb{E}[p_\rho(a - X)^m] = \int_{\mathbb{R}^d} dx\, p_\kappa(x)\, p_\rho(a - x)^m \leq \underbrace{\|p_\kappa\|_{L^\infty(\mathbb{R}^d)}}_{\leq C(d)\,\kappa^{-\frac{d}{2}}} \underbrace{\|p_\rho^m\|_{L^1(\mathbb{R}^d)}}_{\leq C(d)\rho^{-\frac{d}{2}(m-1)}},$$

whence the claim.

By the previous lemma, for all $v < r$, we obtain

$$\|p_{\rho(v,r)}(a - \mathbb{E}^v(X_r))\|_{L^m} \lesssim \kappa(v,r)^{-\frac{d}{2m}} \, \rho(v,r)^{-\frac{d}{2}\left(1-\frac{1}{m}\right)},$$

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so that

$$\begin{aligned}\|\delta A_{s,u,t}\|_{L^m} &= \left\| \int_u^t \left(p_{\rho(s,r)}(a - \mathbb{E}^s(X_r)) - p_{\rho(u,r)}(a - \mathbb{E}^u(X_r))\right) dr \right\|_{L^m} \\ &\lesssim s^{-\alpha_1} |t-s|^{\frac{1}{2}+\epsilon_1},\end{aligned}$$

where $\alpha_1 := \frac{d}{m}\left(H\wedge\frac{1}{2}\right)$ and $\epsilon_1 := \frac{1}{2} - d\left(H\wedge\frac{1}{2}\right)\left(1-\frac{1}{m}\right)$. Since we assumed $d\left(H\wedge\frac{1}{2}\right) < \frac{m}{2(m-1)}$, we have $\alpha_1 \in [0, \frac{1}{2})$ and $\epsilon_1 > 0$. So the SSL applies, and we set $L_t^a := \mathcal{A}_t \in L^m$.

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Let $X_t = B_t + \alpha B_t^H$. Let $f \in L^1(\mathbb{R}^d)$, and $t \geq 0$. Want to show

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By the occupation times formula, and performing the change of variable $b = \sqrt{\lambda} a$, we may rewrite this as

$$\sqrt{\lambda} \int_{-\infty}^{+\infty} f(\sqrt{\lambda} a)\, L^a_t \left(B + \alpha \lambda^{H-\frac{1}{2}} \right) da = \int_{-\infty}^{+\infty} f(b)\, L^{b/\sqrt{\lambda}}_t \left(B + \alpha \lambda^{H-\frac{1}{2}} \right) db.$$

We now use the following result, which can be proven via the SSL:

Lemma

Let $m \in [1, \infty)$. There exists $\eta > 0$ such that

$$\left\| L_t^{b/\sqrt{\lambda}}\left(B + \alpha\lambda^{H-\frac{1}{2}}\right) - L_t^0(B) \right\|_{L^m} \lesssim \lambda^{-\eta}$$

uniformly in $\lambda \geq 1$, locally uniformly in $b \in \mathbb{R}$. Moreover, $\left\| L_t^{b/\sqrt{\lambda}}\left(B + \alpha\lambda^{H-\frac{1}{2}}\right) \right\|_{L^m}$ is bounded uniformly in $\lambda \geq 1$ and $b \in \mathbb{R}$.

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Thanks to these estimates, one easily concludes that, for any $m \in [1, \infty)$,

$$\int_{\mathbb{R}} f(b)\, L_t^{b/\sqrt{\lambda}}\left(B + \alpha\lambda^{H-\frac{1}{2}}\right) db \underset{\lambda\to\infty}{\longrightarrow} \left(\int_{\mathbb{R}} f(b)\, db\right) L_t^0(B)$$

in L^m. Hence the convergence also holds in distribution. This concludes the proof.

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- Scaling limits for additive functionals of more complicated processes (e.g. solutions to SPDEs)?