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Regularization of Multiplicative SDEs through additive noise

CIRM "Pathwise Stochastic Analysis and Applications" - Mar. 10. 2021

Based on joint work with Lucio Galeati

STORM — Stochastics for Time-Space Risk Models
Funded by the Research Council of Norway, project no. 274410, and the University of Oslo

Controlled equations

- Consider the controlled ODE

$$\dot{u}_t = \sigma(u_t)\dot{\beta}_t, \quad u_0 \in \mathbb{R}^d,\ t \in [0, T]. \tag{0.1}$$

  Well known that if σ is Lipschitz w/ linear growth, and $\dot{\beta}$ is continuous, then existence and uniqueness holds.
- Under less restrictive assumptions on σ, we typically loose uniqueness, and only existence holds in general if σ is only continuous and bounded.

ODE vs controlled ODE

- Existence and uniqueness of (0.1) is tightly linked with existence and Uniqueness of the classical ode

$$\dot{y} = \sigma(y_t), \quad t \in \mathbb{R}. \tag{0.2}$$

- Indeed; Let y be a solution to (0.2), and set $u_t = y(\beta_t)$. Given that β is sufficiently nice, by the chain rule u solves (0.1).
- If uniqueness fails for the classical ODE, one can not expect to have uniqueness for the controlled ODE.
- If β is stochastic, with H-continuous sample paths with $H \in (\frac{1}{2}, 1)$, then pathwise techniques may still be used for the Chain rule to apply, and similar arguments holds.

Controlled stochastic equations

- In stochastic analysis, we are interested in the case when $\dot{\beta}_t$ is no longer continuous, but only makes sense as the distributional derivative of some stochastic process $\{\beta_t\}_{t\in[0,T]}$.
- we then typically write

$$\mathrm{d}u_t = \sigma(u_t)\,\mathrm{d}\beta_t \tag{0.3}$$

- Depending on the probabilistic structure of the noise β there are several ways to give meaning to such equations (Itô theory, Young theory, RP theory etc.).
- Again depending on the structure on the noise, we need at least Lipschitz (+) assumption on σ in order to obtain existence and uniqueness.

Regularization by noise for ODEs

- Consider the ODE perturbed by a continuous path $w : [0, T] \to \mathbb{R}^d$

$$x_t = x_0 + \int_0^t \sigma(x_s)\,\mathrm{d}s + w_t \tag{0.4}$$

- We know that perturbations by certain continuous paths in classical ODEs might restore wellposedness, even for distributional σ, see e.g. [1, 4, 6].
- In fact using the framework of non-linear Young theory, existence and uniqueness of (0.4) essentially depends only on the regularity of the *averaged field* given by

$$[0, T] \times \mathbb{R}^d \ni (t, x) \mapsto T^w\sigma(t, x) := \int_0^t \sigma(x + w_s)\,\mathrm{d}s.$$

Regularization by noise for ODEs

- In particular, if $T^w\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$
\begin{aligned}
|T^w\sigma(t,x)| + |\nabla T^w\sigma(t,x)| &\leq C \\
|T^w\sigma(t,x) - T^w\sigma(t,y)| &\leq C|x-y| \\
|\nabla T^w\sigma(t,x) - \nabla T^w\sigma(t,y)| &\leq C|x-y|
\end{aligned}
$$

then there exists a (pathwise) unique solution to (0.4).

Our Goal

- Our goal is to show that a similar regularization by noise phenomena happens in equations with multiplicative noise on the form

$$u_t = u_0 + \int_0^t \sigma(u_t)\, \mathrm{d}\beta_t + w_t \tag{0.5}$$

when $\{\beta_t\}_{t\in[0,T]}$ is a fractional Brownian motion with $H > \frac{1}{2}$.

- That is, we aim to prove that perturbations by a well chosen (Hölder) continuous path $w \in C^\delta$ with $\delta \in (0,1)$ will restore existence and uniqueness of (0.1), even when this fails in the case $w = 0$.
- Heuristically, solution u inherits the regularity $\min(H, \delta)$, and thus the integral is only well defined in the Young sense when $H + \min(\delta, H) > 1$.

Earlier work

- Much work exists on extending the existence and uniqueness results for SDEs with multiplicative noise in the case when $w = 0$.
- Typically extends classical assumptions for existence and uniqueness for ODEs to SDEs (e.g. monotonicity & coercivity assumptions, Osgood condition [7], local-Lipschitz, etc.).
- Difficult to find any work on regularization of multiplicative SDEs by additive noise (with $w \neq 0$).
- We are interested in allowing for distributional σ.

Fractional Brownian motion

- Recall that a centered Gaussian process $\{\beta_t\}_{t\in[0,T]}$ is a fractional Brownian motion with Hurst parameter $H \in (0,1)$ if

$$\mathbb{E}[\beta_t\beta_s] = c_H(t^{2H} + s^{2H} - |t-s|^{2H}), \quad s < t \in [0,T].$$

- Well known that $t \mapsto \beta_t \in C_t^{H-}$ $\mathbb{P}$-a.s..

Notation

- For $p, q \in [1, \infty]$, and $\alpha \in \mathbb{R}$, $B^{\alpha}_{p,q}$ denotes the Besov spaces.
- For $\alpha \in \mathbb{R}$ we use $C^{\alpha}_{x} = C^{\alpha}(\mathbb{R}^d) = B^{\alpha}_{\infty,\infty}$
- For $\gamma \in (0, 1)$ and $\alpha \in \mathbb{R}$, will write $C^{\gamma}_{t} C^{\alpha}_{x} := C^{\gamma}([0, T]; C^{\alpha}(\mathbb{R}^d))$.
- $\mathcal{P}$ will denote a partition of an interval $[a, b] \subset [0, T]$ (specific interval will be clear from context), and $|\mathcal{P}|$ denotes the mesh size.

Young framework and averaged fields

- Assume $w_0 = 0$, and set $\theta = u - w$; observe that θ with $\theta_0 = u_0$ solves

$$\theta_t = u_0 + \int_0^t \sigma(\theta_s + w_s)\, \mathrm{d}\beta_s. \tag{0.6}$$

- If $w \in C_t^\delta$ with $\delta + H > 1$, σ is sufficiently smooth (e.g. C_x^2), and the integral is interpreted in the Young sense, we know there exists a unique solution, and that $\theta \in C_t^{H-}$. We can therefore say that $u \in w + C_t^{H-}$.

Young framework and averaged fields

- If $w \in C^{\delta}$ with $\delta + H > 1$ We have

$$\int_0^t \sigma(\theta_s + w_s)\,\mathrm{d}\beta_s \approx \sum_{[u,v]\in\mathcal{P}} \int_u^v \sigma(\theta_u + w_r)\,\mathrm{d}\beta_r$$

- To know convergence: need to understand regularity of the *multiplicative averaged field*

$$(t,x) \mapsto \Gamma^w\sigma(t,x) := \int_0^t \sigma(x + w_r)\,\mathrm{d}\beta_r.$$

Nonlinear Young integral

Proposition

Let E and V be Banach spaces, and suppose $A \in C^{\gamma}([0,T];C^{\alpha}(E;V))$ and $y \in C^{\rho}([0,T];E)$ such that $\gamma + \alpha\rho > 1$. Then the following limit exists in $C^{\gamma}([0,T];V)$

$$\int_0^t A(\mathrm{d}s, y_s) := \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} A(v, y_u) - A(u, y_u), \tag{0.7}$$

and we say that $t \mapsto \int_0^t A(\mathrm{d}s, y_s)$ is a nonlinear Young integral. Moreover, the mapping $(A, y) \mapsto \int_0^\cdot A(\mathrm{d}s, y_s)$ is continuous on $C^{\gamma}([0,T];C^{\alpha}(E;V)) \times C^{\rho}([0,T];E)$ and linear in A. (see [3, Thm. 2.7]).

Nonlinear Young equations

Proposition

Suppose $A \in C^{\gamma}([0,T]; C^{1+\kappa}(\mathbb{R}^d))$ *with* $\gamma(1+\kappa) > 1$. *Then for any* $x \in \mathbb{R}^d$ *there exists a unique* $\theta \in C^{\gamma}([0,T]; \mathbb{R}^d)$ *which solves*

$$\theta_t = x + \int_0^t A(\mathrm{d}s, \theta_s), \quad \forall t \in [0,T].$$

We call this equation a nonlinear Young equation. (see [3, Thm. 3.12])

Nonlinear Young integral

- Recall that: $T^w\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$T^w\sigma(t,x) = \int_0^t \sigma(x + w_r)\,\mathrm{d}r, \tag{0.8}$$

and suppose $T^w\sigma \in C^\gamma([0,T]; C^\alpha(\mathbb{R}^d))$ with $\gamma(1+\alpha) > 1$.

- Then for any $\theta \in C^\gamma([0,T])$ the following integral is well defined (in a Young sense $\to$ Sewing lemma)

$$\int_0^t \sigma(\theta_s + w_s)\,\mathrm{d}s = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} T^w\sigma(v,\theta_u) - T^w\sigma(u,\theta_u). \tag{0.9}$$

- If σ is continuous the integral agrees with the classical Riemann integral.

Regularity of $\Gamma^w \sigma$

- Consider again the (stochastic) multiplicative averaged field

$$\Gamma^w \sigma(t,x) = \int_0^t \sigma(x + w_r)\, \mathrm{d}\beta_r, \quad (t,x) \in [0,T] \times \mathbb{R}^d. \quad (0.10)$$

- *Would knowledge of the regularity of $(t,x) \mapsto T^w \sigma(t,x)$ give us knowledge of the pathwise regularity of the stochastic field $(t,x) \mapsto \Gamma^w \sigma(t,x)$?*
- If so, then we could define the integral appearing in (0.6) in a pathwise way by application again of the sewing lemma.
- More precisely: if we can prove that $\Gamma^w \sigma \in C^{\gamma'}([0,T]; C^\alpha(\mathbb{R}^d))$ $\mathbb{P}$-a.s. for some $\gamma' > \frac{1}{2}$ and $\alpha \geq 2$, then pathwise existence and uniqueness holds for (0.6) (as a nonlinear Young equation).

Stochastic estimates

- M. Hairer and X-M. Lie (2020) [5]: For some $f : [0,T] \to \mathbb{R}^d$ with f_0 being $\mathcal{F}_0$ measurable, suppose $\| \int_0^\cdot f_r \, \mathrm{d}r \|_\gamma \in L^q(\Omega)$ for some $q > 2$. Then for any $p \in [2, q)$ there exists a constant $C > 0$ such that

$$\left\| \int_s^t f_r \, \mathrm{d}\beta_r \right\|_{L^p(\Omega)} \leq C \left\| \left\| \int_0^\cdot f_r \, \mathrm{d}r \right\|_\gamma \right\|_{L^q(\Omega)} |t-s|^{H+\gamma-1}. \quad (0.11)$$

- Note that if w is deterministic, $T^w\sigma(t,x) = \int_0^t f_r \, \mathrm{d}r$ with $f_r = \sigma(x + w_r)$ being deterministic, so $\| \int_0^\cdot f_r \, \mathrm{d}r \|_\gamma \in L^q(\Omega)$ for any $q > 2$.
- !(0.11) does not give explicit regularity requirements on w!

Regularity of multiplicative averaged field

Theorem

If $w \in C_t^{\delta}$ and $\delta + H > 1$, then for any $b \in C_c^{\infty}(\mathbb{R}^d)$, the multiplicative averaged field $\Gamma^w \sigma$, defined as a Young integral, is a random field from $[0, T] \times \mathbb{R}^d \to \mathbb{R}^d$. The definition extends continuously in a unique way to any $(\sigma, w) \in \mathcal{D}(\mathbb{R}^d) \times C([0, T]; \mathbb{R}^d)$ such that $T^w \sigma \in C_t^{\gamma} C_x^{\eta}$ for some $\gamma > 1 - H$, $\eta > 0$. In that case

$$\Gamma^w \sigma \in L^p(\Omega; C_t^{\gamma'} C_x^{\eta'}), \ \forall p < \infty, \gamma' < \gamma + H - 1, \eta' < \eta.$$

Furthermore, for any two pairs (w^i, σ^i), $i = 1, 2$, we have

$$\mathbb{E}\left[\| \Gamma^{w^1} \sigma^1 - \Gamma^{w^2} \sigma^2 \|_{\gamma', \eta'}^p \right] \lesssim \| T^{w^1} \sigma^1 - T^{w^2} \sigma^2 \|_{\gamma, \eta}^p.$$

Strategy for proof:

- When $\delta + H > 1$ and $\sigma \in C_c^\infty(\mathbb{R}^d)$ the integral may be constructed analytically (as a Young integral; construction would hold for any $\beta \in C_t^{H-}$, not only fBm).
- By lemma of Hairer and Li $\rightarrow$ may extend to more general (w, σ) such that $T^w\sigma$ is sufficiently regular.
- A modified Garsia-Rodemich-Rumsey inequality allows us to give the conclusion.

Regularizing paths

- we say that w is a *regularizing path* if for some $\alpha \in \mathbb{R}$ and function $\sigma \in C_x^\alpha := B^\alpha_{\infty,\infty}$ there exists a $\rho > 0$ such that $T^w\sigma \in C_t^\gamma C_x^{\alpha+\rho}$ for some $\gamma > \frac{1}{2}$.
- What type of continuous paths $w : [0, T] \to \mathbb{R}^d$ is regularizing?
- Example: Almost all sample paths of a fractional Brownian motion is regularizing with $\rho = \frac{1}{2H} - \epsilon$ (for any $\epsilon > 0$). [1]
- $\exists$ a class of continuous Gaussian processes, such that for almost all sample paths, and any $\sigma \in \mathcal{D}$ the averaged field $T^w\sigma \in C^\gamma([0, T]; C^n(\mathbb{R}^d))$ for some $\gamma > \frac{1}{2}$ and any $n \in \mathbb{N}$. [6]

Regularizing paths

- Heuristically, we observe that the more irregular trajectories of the stochastic process W is, the better regularization effect is seen in $T^w\sigma$.
- Regularity of $T^w\sigma$ has strong connection with regularity of the local time associated with the path w. Note that

$$T^w\sigma(t,x) = (\sigma * L_t^w)(x), \tag{0.12}$$

where L^w is the denotes the local time, given as the density associated to the occupation measure

$$\mu_t(A) := \{s \le t; w_s \in A\}, \quad A \in \mathcal{B}(\mathbb{R}^d). \tag{0.13}$$

Main Theorem 1

Theorem

Let $H \in (\frac{1}{2}, 1)$*,* σ *be a distribution such that* $T^w\sigma \in C_t^\gamma C_x^2$ *for some* $\gamma \in (\frac{3}{2} - H, 1)$*. Then path-by-path existence and uniqueness holds for*

$$\theta_t = x + \int_0^t \Gamma^w \sigma(\mathrm{d}s, \theta_s), \quad \forall t \in [0, T], \tag{0.14}$$

where the integral is defined for a.a. $\omega \in \Omega$ *by*

$$\int_0^t \Gamma^w \sigma(\mathrm{d}s, \theta_s)(\omega) = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v]} \Gamma^w \sigma(v, \theta_u)(\omega) - \Gamma^w \sigma(u, \theta_u)(\omega). \tag{0.15}$$

Main Theorem 2

Theorem

Suppose β is a sample path of a fBm with $H > \frac{1}{2}$. Let σ be a compactly supported distribution in C_x^α and w be a sample path of an independent fBm with Hurst parameter $\delta \in (0,1)$, and assume

$$\alpha > 2 - \frac{1}{\delta}(H - \frac{1}{2}). \tag{0.16}$$

Then for any $u_0 \in \mathbb{R}^d$, existence and path-by-path uniqueness holds for the equation

$$u_t = u_0 + \int_0^t \sigma(u_s)\,\mathrm{d}\beta_s + w_t \tag{0.17}$$

(i.e. $u = \theta + w$).

Strategy of proof

- Know from [4] that for $\sigma \in C_x^\alpha$ and W being fBm with Hurst parameter $\delta \in (0,1) \implies T^W\sigma \in C_t^\gamma C_x^{\alpha+\kappa}$ for any $\gamma \in (\frac{1}{2}, 1]$ and $\kappa = \frac{1-\gamma}{2\delta}$ $\mathbb{P}$-a.s..
- Choose $w = W(\omega) \in C^{\delta-}$ such that $T^w\sigma \in C_t^\gamma C_x^{\alpha+\kappa}$
- From stochastic estimate, we have $\Gamma^w\sigma \in C_t^{\gamma'} C_x^{\alpha+\kappa'}$ for

$$\gamma' < \gamma + H - 1 \quad \textit{and} \quad \kappa' < \kappa. \tag{0.18}$$

- In order to apply NLY theory; need $\gamma' > \frac{1}{2}$ and $\alpha + \kappa' \geq 2$, which yields the claimed requirement.

Conclusion

- Perturbations by noise may regularize multiplicative SDEs.
- Did not use the Young requirement that $H + \delta > 1$!
- Simply extended to time dependent vector field σ, and equations with drift, e.g.

$$u_t = u_0 + \int_0^t b(s, u_s)\,\mathrm{d}s + \int_0^t \sigma(s, u_s)\,\mathrm{d}\beta_s + w_t. \tag{0.19}$$

Conclusion

- Removing the noise would be challenging: Similar to zero noise limit for ODEs.
- Extensions to rougher driving noise is challenging: Possibly need non-linear rough path theory and might not yield good regularization.
- if β is a B.M., Itô isometry may be used to connect (deterministic) averaged fields, and multiplicative averaged fields. Would result in Sobolev regularity requirements for σ of at least $\sigma \in H^2$ or similar.
- But already know the equation is well-posed for $\sigma \in H^1$ (see [2]).
- Rougher fBm seems to be worse...

The end! Thanks! I

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The end! Thanks! II

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