Paracontrolled calculus and regularity structures

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9 March, 2021

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Two approaches to singular PDEs

Singular SPDEs contain ill-posed multiplications, e.g., generalized KPZ equation

$$\partial_t h = \partial_x^2 h + \underbrace{f(h)}_{\frac{1}{2}-}\underbrace{(\partial_x h)}_{-\frac{1}{2}-}{}^2 + \underbrace{g(h)}_{\frac{1}{2}-}\underbrace{\xi}_{-\frac{3}{2}-}$$

Multiplication $C^\alpha \times C^\beta \to C^{\alpha\wedge\beta}$ is well-posed iff $\alpha+\beta>0$.

Two approaches

- Regularity structure (Hairer, 2014)
- Paracontrolled calculus (Gubinelli-Imkeller-Perkowski, 2015)
  $\to$ High order PC (Bailleul-Bernicot, 2019)

The two approaches are different but believed to be equivalent.

Micro vs Macro

Both of RS and PC are extensions of the rough path theory for SDEs

$$dX = F(X)dB.$$

- RS provides a microscopic (pointwise) description

$$X_t - X_s = F(X_s)(B_t - B_s) + O(|t-s|^{1-}).$$

- PC provides a macroscopic (spectral) description

$$X = F(X) \varolessthan B + (C^{1-}).$$

$\varolessthan$: Bony's paraproduct

$$f \varolessthan g = \sum_{i<j-1} \rho(2^{-i}\nabla)f \cdot \rho(2^{-j}\nabla)g,$$

where $\rho(2^{-i}\cdot)$ denotes a dyadic decomposition of 1.

Our aim is to show

microscopic description $\Leftrightarrow$ macroscopic description

Main result (rough)

We obtained the equivalence between the two descriptions.

Rough path theory	RS		PC
Rough path	Model	$\Leftrightarrow$ [1]	Pararemainders
Controlled path	Modelled distribution	$\Leftrightarrow$ [2]	Paracontrolled distribution
Stochastic integral	[Chandra-Hairer, 2016]	Future work	No systematic theory

- [1] Bailleul-H, 2020
- [2] Bailleul-H, 2019 (on arXiv)

Related researches

- Martin-Perkowski, 2020 : paraproducts on RS.
- Tapia-Zambotti, 2020 : similar result for the branched rough paths.

Concrete regularity structure

Branched RP is a continuous path from $[0,T]$ to Butcher group, which is a character group on Connes-Kreimer algebra.

Hopf algebra H = "Jointing trees" + "Spliting a tree"
= product $(\cdot : H \otimes H \to H)$ + coproduct $(\Delta : H \to H \otimes H)$.

Definition

A concrete regularity structure (T^+, T) *consists of*

1. *Connected graded Hopf algebra* $T^+ = \bigoplus_{\alpha \in A^+} T^+_\alpha$.

$$A^+ \subset [0,\infty) \quad \textit{loc. fin.}, \quad \dim T^+_0 = 1, \quad \dim T^+_\alpha < \infty,$$
$$T^+_{\alpha_1} \cdot T^+_{\alpha_2} \subset T^+_{\alpha_1+\alpha_2},$$
$$\Delta^+ : T^+ \to T^+ \otimes T^+, \quad \Delta^+ T^+_\alpha \subset \oplus_{\alpha_1+\alpha_2=\alpha}(T^+_{\alpha_1} \otimes T^+_{\alpha_2}).$$

2. *Graded right comodule* $T = \bigoplus_{\beta \in A} T_\beta$.

$$A \subset \mathbb{R} \quad \textit{loc. fin.}, \quad \inf A > -\infty, \quad \dim T_\beta < \infty,$$
$$\Delta : T \to T \otimes T^+, \quad \Delta T_\beta \subset \oplus_{\beta_1+\beta_2=\beta}(T_{\beta_1} \otimes T^+_{\beta_2}).$$

Some remarks

Polynomial regularity structure is an easy example of RS.

- $T^+ = T = \mathbb{R}[X_1, \dots, X_d]$.
- $X^k := \prod_{i=1}^d X_i^{k_i}$, where $k = (k_i)_{i=1}^d \in \mathbb{N}^d$.
- Product $X^k \cdot X^\ell = X^{k+\ell}$.
- Coproduct $\Delta X^k = \sum \binom{k}{\ell} X^\ell \otimes X^{k-\ell}$.

Character group

Since T^+ is a Hopf algebra, the set G of algebra morphisms $g : T^+ \to \mathbb{R}$ forms a group with

- Product $(g_1 * g_2)(\tau) := (g_1 \otimes g_2)\Delta\tau$.
- Inverse $g^{-1} := g \circ S$, S is the antipode of T^+.

$G \curvearrowright T$ by

$$\hat{g}(\tau) := (\mathrm{id} \otimes g)\Delta\tau.$$

Original RS by Hairer consists of the pair (T, G).

Models

Rough path theory	RS
Rough path	Model
Controlled path	Modelled distribution
Sewing lemma	Reconstruction theorem

Definition (Model)

The space $\mathcal{M}$ *consists of the pair* $M = (g, \Pi)$ *such that*

- $g : \mathbb{R}^d \ni x \mapsto g_x \in G$ *is a continuous map such that*

$$g_{yx}(\tau) := (g_y * g_x^{-1})(\tau) = O(|y-x|^{\alpha}), \quad \tau \in T_\alpha^+.$$

- $\Pi : T \to \mathcal{S}'(\mathbb{R}^d)$ *is a bounded operator such that*

$$\Pi_x \tau(y) := (\Pi \otimes g_x^{-1})\Delta\tau(y) = O(|y-x|^{\beta}), \quad \tau \in T_\beta.$$

Modelled distributions

Definition (Modelled distribution)

For $\gamma \in \mathbb{R}$ and any $M = (g, \Pi) \in \mathcal{M}$, the space $\mathcal{D}^\gamma(g)$ consists of all maps $f : \mathbb{R}^d \to T_{<\gamma} := \bigoplus_{\alpha<\gamma} T_\alpha$ such that

$$\big(f(y) - \widehat{g_{yx}} f(x)\big)_{T_\alpha} = O(|y-x|^{\gamma-\alpha}), \quad \alpha < \gamma.$$

Reconstruction operator is a bounded operator $\mathcal{R}^M : \mathcal{D}^\gamma(g) \to \mathcal{S}'(\mathbb{R}^d)$ such that

$$\mathcal{R}^M f(y) = \big(\Pi_x f(x)\big)(y) + O(|y-x|^\gamma), \quad f \in \mathcal{D}^\gamma(g).$$

Theorem (Hairer, 2014 & Caravenna-Zambotti, 2020)

- *If $\gamma > 0$, the operator $\mathcal{R}^M$ uniquely exists.*
- *If $\gamma < 0$, the operator $\mathcal{R}^M$ exists but it is not unique.*
- *If $\gamma = 0$, the operator $\mathcal{R}^M$ does not exists (logarithmic estimate is needed instead of boundedness).*

From RS to PC

We prove

RS		PC
Model	$\Rightarrow$	Pararemainders
Modelled distribution	$\Rightarrow$	Paracontrolled distribution

Notations

- Each $\tau \in T_\alpha^{(+)}$ is said to be of homogeneity α. We write

$$|\tau| = \alpha.$$

- Fix a homogeneous basis $\mathcal{B}^{(+)}$ of $T^{(+)}$. For any $\tau, \sigma \in \mathcal{B}^{(+)}$, we define the element $\tau/\sigma \in T^+$ by

$$\Delta^{(+)}\tau = \sum_{\sigma \in \mathcal{B}^{(+)}} \sigma \otimes (\tau/\sigma).$$

  In CK algebra and BHZ (Brined-Hairer-Zamotti, 2019) algebra, σ is a subtree of τ and τ/σ is a quotient graph.

Model $\Rightarrow$ Pararemainders

For technical reasons, we consider the Hölder space with polynomial weights. We omit the details here.

Theorem (Bailleul-H, 2020)

Let $M = (g, \Pi) \in \mathcal{M}$. There exist continuous linear maps

$$[\cdot]^g : T^+ \to C(\mathbb{R}^d), \quad [\cdot]^M : T \to \mathcal{S}'(\mathbb{R}^d).$$

such that

- *For any $\tau \in T^+_\alpha$, one has $[\tau]^g \in C^\alpha$, and*

$$g(\tau) = \sum_{\eta \in \mathcal{B}^+,\, |\eta| < \alpha} g(\tau/\eta) \otimes [\eta]^g + [\tau]^g.$$

- *For any $\sigma \in T_\beta$, one has $[\sigma]^M \in C^\beta$, and*

$$\Pi\sigma = \sum_{\zeta \in \mathcal{B},\, |\zeta| < \beta} g(\sigma/\zeta) \otimes [\zeta]^M + [\sigma]^M.$$

Modelled distribution ⇒ Paracontrolled distribution

Proposition (Bailleul-H, 2020)

Let $\gamma \in \mathbb{R}$ *and* $M = (g, \Pi) \in \mathcal{M}$*. For any modelled distribution*

$$f = \sum_{\tau \in \mathcal{B},\, |\tau| < \gamma} f_\tau \tau \in \mathcal{D}^\gamma(g),$$

one has

$$f_\sigma = \sum_{\tau \in \mathcal{B},\, |\sigma| < |\tau| < \gamma} f_\tau \otimes [\tau/\sigma]^g + [f_\sigma]^g, \quad \sigma \in \mathcal{B},$$

with $[f_\sigma]^g \in C^{\gamma - |\sigma|}$*. Moreover, the reconstruction* $\mathcal{R}^M f$ *is of the form*

$$\mathcal{R}^M f = \sum_{\tau \in \mathcal{B},\, |\tau| < \gamma} f_\tau \otimes [\tau]^M + [f]^M,$$

where $[f]^M \in C^\gamma$*.*

These formulas give an algebraic meaning to the paracontrolled systems (Gubinelli-Imkeller-Perkowski, 2015 and Bailleul-Bernicot, 2019).

From PC to RS

Recall

$$g(\tau) = \sum_{\eta\in\mathcal{B}^+,\, |\eta|<|\tau|} g(\tau/\eta) \otimes [\eta]^g + [\tau]^g,$$

$$\Pi\sigma = \sum_{\zeta\in\mathcal{B},\, |\zeta|<|\sigma|} g(\sigma/\zeta) \otimes [\zeta]^M + [\sigma]^M,$$

$$f_\sigma = \sum_{\tau\in\mathcal{B},\, |\sigma|<|\tau|<\gamma} f_\tau \otimes [\tau/\sigma]^g + [f_\sigma]^g.$$

To recover the original (g,Π) and f from $[\tau]^g, [\sigma]^M$, and $[f_\sigma]^g$, we need some additional (but harmless) assumptions.

Polynomials and derivatives

Assumption

Let $\mathcal{B}^{(+)}$ *be a homogeneous basis of* $T^{(+)}$.

1. *There exists a "generating" set* $\mathcal{G}_\circ^+ \subset \mathcal{B}^+$ *such that, each element* $\tau \in \mathcal{B}^+$ *is uniquely represented as*

$$\tau = X^k \prod_{n=1}^{N} (\tau_n / X^{k_n}),$$

where $k, k_1, \ldots, k_N \in \mathbb{N}^d$ *and* $\tau_1, \ldots, \tau_n \in \mathcal{G}_\circ^+$. *Moreover, the splitting map* Δ^+ *admits some inductive structure (e.g. scale of the graph).*

2. *There exists a subset* $\mathcal{B}_\bullet \subset \mathcal{B}$ *such that, each element* $\sigma \in \mathcal{B}$ *is uniquely represented as*

$$\sigma = X^k \eta,$$

where $k \in \mathbb{N}^d$ *and* $\eta \in \mathcal{B}_\bullet$.

3. *Any nonpolynomial element of* $\mathcal{B}^{(+)}$ *has noninteger homogeneity.*

Graphical meanings

In BHZ algebra,

- $\mathcal{B}_\bullet$: all strongly conform trees with $\mathfrak{n} = 0$ at those roots.
- $\mathcal{G}_\circ^+$: all "planted" trees with $\mathfrak{e} = 0$ at the edges leaving from their roots.

Assumptions on models

In what follows, we consider only the models (g, Π) such that

$$g_x(X^k) = x^k, \quad \Pi(X^k \eta)(x) = x^k (\Pi\eta)(x).$$

These are natural assumptions on polynomial elements.

Model $\Leftarrow$ Pararemainders

Theorem (Bailleul-H, 2019)

Subfamilies

$$\{[\tau]^g \in C^{|\tau|}\,;\, \tau \in \mathcal{G}_\circ^+\}, \quad \{[\sigma]^M \in C^{|\sigma|}\,;\, \sigma \in \mathcal{B}_\bullet,\ |\sigma| < 0\}.$$

are sufficient to recover the original model $M = (g, \Pi)$*. This inverse map is continuous, so one obtains a* homeomorphism

$$\mathcal{M} \simeq \prod_{\tau \in \mathcal{G}_\circ^+} C^{|\tau|} \times \prod_{\sigma \in \mathcal{B}_\bullet,\, |\sigma|<0} C^{|\sigma|}$$

cf. Admissible models (by Hairer) are recovered by only

$$\{[\sigma]^M \in C^{|\sigma|}\,;\, \sigma \in \mathcal{B}_\bullet,\ |\sigma| < 0\},$$

since then T^+ and T are interwined.

Additional assumptions

Assumption

For any $\tau \in \mathcal{B}_\bullet$, its coproduct $\Delta\tau$ does not have components of the form

$$\sigma \otimes X^k$$

with $k \neq 0$.

BHZ algebra does not seem to satisfy this assumption. Indeed,

$$\Delta I(X\Xi) = I(\Xi) \otimes X + \cdots .$$

However,

Proposition (Bailleul-H, 2019)

There is another basis of BHZ algebra which satisfies the above assumption.

We exchange $\mathfrak{n}$-decoration for the convolution with $x^k \partial^\ell K_{\mathfrak{t}}(x)$ ($K_{\mathfrak{t}}$ is the integral kernel of type $\mathfrak{t}$).

Modelled distribution $\Leftarrow$ Paracontrolled distribution

Theorem (Bailleul-H, 2019)

Assume that $\gamma \neq 0$ and $\gamma - |\tau| \notin \mathbb{N}$ for any $\tau \in \mathcal{B}$. Then a subfamily

$$\{[f_\sigma]^g \,;\, \sigma \in \mathcal{B}_\bullet,\, |\sigma| < \gamma\}$$

is sufficient to recover the original modelled distribution $f \in \mathcal{D}^\gamma(g)$. This inverse map is continuous, so one obtains a homeomorphism

$$\mathcal{D}^\gamma(g) \simeq \prod_{\tau \in \mathcal{B}_\bullet,\, |\tau| < \gamma} C^{\gamma - |\tau|}.$$