Averaging principle for fast-slow system driven by mixed fractional Brownian rough path

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Aim of the talk

♠ We consider the following fast-slow system of stochastic equations:

$$\begin{cases} \mathrm{d}X_t^\varepsilon = f(X_t^\varepsilon, Y_t^\varepsilon)\,\mathrm{d}t + \sigma(X_t^\varepsilon)\,\mathrm{d}B_t^H, \\ \mathrm{d}Y_t^\varepsilon = \frac{1}{\varepsilon} g(X_t^\varepsilon, Y_t^\varepsilon)\,\mathrm{d}t + \frac{1}{\sqrt{\varepsilon}} h(X_t^\varepsilon, Y_t^\varepsilon)\,\mathrm{d}W_t. \end{cases} \tag{1}$$

Here, $0 < \varepsilon \ll 1$ is a small parameter,
the initial value (x_0, y_0) is deterministic and indep of ε,
(W_t) is standard BM, and (B_t^H) is FBM with $H \in (\frac{1}{3}, \frac{1}{2}]$.
(the two processes are indep.)

♣ First, we formulate this system as RDE driven by the RP lift of (B^H, W) i.e., "mixed fractional BRP".

♠ Impose suitable assumptions on g, h so that the fast component Y becomes ergodic.

♡ Prove a (strong) averaging principle for X^ε, that is, as $\varepsilon \searrow 0$, X^ε converges to a natural limit process $\bar{X}$ in L^1-sense. (The limit satisfies the "averaged" equation.)

♢ To our knowledge, this is the first averaging result for fast-slow system in the framework of RP theory.
(Note: Different problems are also called "averaging".)

Background

AP for fast-slow systems has a long history and seems still quite active.
According to Freidlin-Wentzell's book, Soviet mathematicians did lots of works for this kind of averaging problems.

- ODE case $\longrightarrow$ Bogolyubov, Volosov, Neishtadt, Anosov, etc.
- SDE case $\longrightarrow$ Khas'minskii, Freidlin, Veretennikov, etc.

Note: This speaker is not an expert of AP.
So, the list is probably far from complete.

(Review: standard SDE case)

A typical formulation for the SDE case:

$$\begin{cases} \mathrm{d}X_t^\varepsilon = f\left(X_t^\varepsilon, Y_t^\varepsilon\right)\mathrm{d}t + \sigma\left(X_t^\varepsilon, Y_t^\varepsilon\right)\mathrm{d}B_t, \\ \mathrm{d}Y_t^\varepsilon = \frac{1}{\varepsilon} g\left(X_t^\varepsilon, Y_t^\varepsilon\right)\mathrm{d}t + \frac{1}{\sqrt{\varepsilon}} h\left(X_t^\varepsilon, Y_t^\varepsilon\right)\mathrm{d}W_t. \end{cases}$$

Here, (B_t) and (W_t) are two independent BM's.

The AP is a limit thorem for the slow component X^ε.
There are two types of the averaging principle.

- Weak AP $\longrightarrow$ limit in probability,
- Strong AP $\longrightarrow$ limit in L^1 or L^p $(1 \leq p < \infty)$.

Concerning this, the following are well-known among experts:

(A) For the diffusion coefficient σ in front of dB, there are two types of setting, namely $\sigma(X, Y)$ and $\sigma(X)$.
Weak AP was proved for $\sigma(X, Y)$-type, but
strong AP was proved for $\sigma(X)$-type only.

(B) In fact, a counterexample is known.
Givon ('07) find a fast-slow system of $\sigma(X, Y)$-type for which weak AP holds, but strong AP fails.

(Review: the case of FBM with $H > 1/2$)
Formally, the same fast-slow system:

$$\begin{cases} \mathrm{d}X_t^\varepsilon = f\left(X_t^\varepsilon, Y_t^\varepsilon\right)\mathrm{d}t + \sigma\left(X_t^\varepsilon, Y_t^\varepsilon\right)\mathrm{d}B_t^H, \\ \mathrm{d}Y_t^\varepsilon = \frac{1}{\varepsilon} g\left(X_t^\varepsilon, Y_t^\varepsilon\right)\mathrm{d}t + \frac{1}{\sqrt{\varepsilon}} h\left(X_t^\varepsilon, Y_t^\varepsilon\right)\mathrm{d}W_t. \end{cases}$$

Here, (W_t) is standard BM and (B_t^H) is FBM with $H \in (\frac{1}{2}, 1)$.
(the two processes are indep.)

- X.M. Li - Hairer ('20). Weak AP for $\sigma(X, Y)$-type.
  Young integration is used for X-component.
- Pei-I.-Xu ('20+). Strong AP for $\sigma(X)$-type.
  Fractional calculus generalization of Young integral in Zähle's, Nualart-Guerra-Rascanu's way for X-component.

(Natural Question) What happens for smaller H?
When $\frac{1}{4} < H \leq \frac{1}{2}$, FBRP exists and RP theory is available.

(Our Main Result) When $\frac{1}{3} < H \leq \frac{1}{2}$, we prove the strong AP for $\sigma(X)$-type (convergence in L^1).

(Method) • Carry out the classical Khas'minskii discretization method in the framework of RP theory.
• Use Nualart-Hu's fractional calculus approach to RP theory. (But, this does not seem essential.)

Mixed fractional Brownian Rough Path

- Assume $\frac{1}{3} < H \leq \frac{1}{2}$ and write $B^H = B$. Then, natural RP lift of d'-dim BM W and d-dim FBM B exists: (W, W^2) and (B, B^2).
- A natural lift of $Z := (B, W)$ should be

$$Z^2_{st} = \begin{pmatrix} B^2_{st} & \int_s^t (B_u - B_s) \otimes \mathrm{d}W_u \\ \star & W^2_{st} \end{pmatrix},$$

where $\star = W^1_{st} \otimes B^1_{st} - \int_s^t \mathrm{d}W_u \otimes (B_u - B_s)$. (Integrals are Itô).
- FS system (1) is understood as RDE driven by (Z, Z^2).

  $\rightsquigarrow$ "dW" in (1) is something like Stratonovich.

[*cf*] Diehl-Oberhauser-Riedel '15, Neuenkirch-Shalaiko '15, in which (B, B^2) is deterministic.

Assumptions on coefficients 1

Redisplay our FS sytem:

$$\begin{cases} \mathrm{d}X_t^\varepsilon = f\left(X_t^\varepsilon, Y_t^\varepsilon\right)\mathrm{d}t + \sigma\left(X_t^\varepsilon\right)\mathrm{d}B_t, \\ \mathrm{d}Y_t^\varepsilon = \dfrac{1}{\varepsilon}g\left(X_t^\varepsilon, Y_t^\varepsilon\right)\mathrm{d}t + \dfrac{1}{\sqrt{\varepsilon}}h\left(X_t^\varepsilon, Y_t^\varepsilon\right)\mathrm{d}W_t. \end{cases}$$

The coefficients $f: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m, \quad g: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n,$ $\sigma: \mathbb{R}^m \to \mathbb{R}^m \otimes \mathbb{R}^d, \quad h: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^{d'}$.

- (H1) f is a locally Lipschitz continuous vector field with at most linear growth, $\sigma \in C_b^4(\mathbb{R}^m; \mathbb{R}^m \otimes \mathbb{R}^d)$.
- (H2) g is a locally Lipschitz continuous vector field with at most linear growth, $h \in C_b^4(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}^n \otimes \mathbb{R}^{d'})$.

(Remark) It is important to allow g to have linear growth.
(f is assumed to be bounded in the next step.)
Otherwise, it would be very hard to find meaningful examples.

[Riedel-Scheutzow '17] "RDE with unbounded drift term"
Thanks to this result, the above FS system of RDEs have a unique solution under (H1) and (H2).
(The price to pay is that C_b^4-condition on σ, h, not C_b^3.)

So, for each realization of (Z, Z^2), there exists a unique solution $(X^\varepsilon, Y^\varepsilon)$, which we will study.

Assumptions on coefficients 2

- (H3) $f \in C_b^1(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}^m)$.
- (H4) $\exists L > 0$, $\exists \beta_i > 0, i = 1, 2$, such that

$$\begin{aligned} 2\langle \phi - \tilde{\phi}, \tilde{g}(\xi, \phi) - \tilde{g}(\xi, \tilde{\phi}) \rangle \quad &+ \quad |h(\xi, \phi) - h(\xi, \tilde{\phi})|^2 \\ &\leq \quad -\beta_1 |\phi - \tilde{\phi}|^2, \\ 2\langle \phi, \tilde{g}(\xi, \phi) \rangle + |h(\xi, \phi)|^2 \quad &\leq \quad -\beta_2 |\phi|^2 + L|\xi|^2 + L \end{aligned}$$

for any $\xi \in \mathbb{R}^m$ and $\phi, \tilde{\phi} \in \mathbb{R}^n$, where

$$\tilde{g}(\xi, \phi) := g(\xi, \phi) + \frac{1}{2} \sum_{\bar{l}=1}^{n} \sum_{\bar{j}=1}^{d'} \mathcal{D}_h^{(\bar{j})} h_{\bar{l}, \bar{j}}(\xi, \phi), \ \mathcal{D}_h^{(\bar{j})} = \sum_{\bar{l}=1}^{n} h_{\bar{l}, \bar{j}}(\cdot, \cdot) \partial_{\phi_{\bar{l}}}.$$

- Consider the following "frozen SDE" for $\forall$ fixed $\xi \in \mathbb{R}^m$:

$$\mathrm{d}Y_t^{\xi,\phi} = \tilde{g}(\xi, Y_t^{\xi,\phi})\mathrm{d}t + h(\xi, Y_t^{\xi,\phi})\mathrm{d}^{\mathrm{I}}W_t, \quad Y_0^{\xi,\phi} = \phi \in \mathbb{R}^n \quad (2)$$

Here, $\int \cdots \mathrm{d}^{\mathrm{I}}W$ stands for the usual Itô integral.

- Under (H4), $\exists!$ invariant prob. meas. μ^ξ for above SDE (2).
- Define the "averaged RDE" by

$$\mathrm{d}\bar{X}_t = \bar{f}(\bar{X}_s)\mathrm{d}t + \sigma(\bar{X}_t)\mathrm{d}B_t, \qquad \bar{X}_0 = x_0, \qquad (3)$$

where

$$\bar{f}(\xi) = \int_{\mathbb{R}^n} f(\xi, \phi)\mu^\xi(\mathrm{d}\phi), \quad \xi \in \mathbb{R}^m.$$

(Fast variable of f is "averaged out.")

Our Main Result

(Strong-type averaging principle):
As $\varepsilon \searrow 0$, X^ε converges to the sol. $\bar{X}$ of the averaged equation in sup-norm in L^1.

Theorem 1

Let $\frac{1}{3} < H \le \frac{1}{2}$ and (Z, Z^2) be the natural rough path lift of $(B_t, W_t)_{t\in[0,T]}$. Assume that f, σ, g, h satisfy (H1)-(H4). Then, we have

$$\lim_{\varepsilon\to 0} \mathbb{E}[\|X^\varepsilon - \bar{X}\|_\infty] = 0.$$

Here, $\|\cdot\|_\infty$ denotes the supremum norm over $[0, T]$ and X^ε and $\bar{X}$ denote the first level paths of the slow component of (1) and (3), respectively.

- Examples of f, σ, g, h do exist since we do not assume boundedness of g. (Of course, (H4) is the problem.)
- For instance, the following g, h satisfy (H4).

  - When $d = d' = m = n = 1$,

$$g(\xi, \phi) = \xi - 8\phi \text{ and } h(\xi, \phi) = \sin\xi + \sin\phi$$

  - Let $g(\xi, \phi) = -A(\xi)\phi$, where A is a bounded, positive, C_b^1-function in ξ, which is also bounded away from zero. If C_b^2-norm of h is sufficiently small, then these g and h satisfy (H4).
  - Maybe, more....

Framework of our proof

- We basically use Hu-Nualart's fractional calculus approach to RP theory. ($\nexists$ no special meaning here)
- $\exists$ Extensions of this approach: Yu Ito (higher level case), Garrido-Atienza and Schmalfuss '18 (with drift) among others.
- We do not show their formulation in our slides, because these equations are quite long. (Sorry!)
- Other formulations of RP theory (e.g., controlled path theory) are probably OK, too. But, not confirmed yet.

Simple Observations

♣ Fast component Y^ε actually satisfies the following Itô integral "equation":

$$Y_t^\varepsilon = \frac{1}{\varepsilon}\int_0^t \tilde{g}\left(X_s^\varepsilon, Y_s^\varepsilon\right) \mathrm{d}s + \frac{1}{\sqrt{\varepsilon}}\int_0^t h\left(X_s^\varepsilon, Y_s^\varepsilon\right) \mathrm{d}^{\mathrm{Ito}} W_s.$$

We mimick Neuenkirch-Shalaiko '15, etc.
(For this part, we use controlled path theory.)

♣♣ Now integral on RHS is of Itô-type, we can do a typical computation for ergodicity/∃! invariant distribution for SDEs under dissipative-type condition (H4).

♠ Let's observe that slow component X^ε is not so "strongly coupled" with Y^ε.

♠♠ For any usual deterministic continuous path $y = (y_t)$, the following RDE driven by (B, B^2) makes sense and well-posed since we assumed f is of C_b^1 in (H3).

$$\mathrm{d}\hat{X}_t^y = f\left(\hat{X}_t^y, y_t\right) \mathrm{d}t + \sigma\left(\hat{X}_t^y\right) \mathrm{d}B_t$$

We have the same estimates for $\hat{X}^y$ as for the standard RDE with C_b^3-coefficient and these estimates do not depend on y.

♠♠♠ If $y =$ first level of $Y^\varepsilon \quad \Longrightarrow \quad \hat{X}^y = X^\varepsilon$.
So, estimating X^ε is not very difficult (and these estimates do not depend on ε).

Discretization Method

We carry out Khas'minskii's discretization method in the RP framework.

♣ Take $\delta \in (0, 1)$ so that $0 < \varepsilon < \delta$ and $\varepsilon/\delta \to 0$ as $\varepsilon \searrow 0$, e.g. $\delta := \varepsilon\sqrt{-\log \varepsilon}$.

♣ Divide the time interval $[0, T]$ into subintervals of equal length δ. For $s \in [0, T]$, $s(\delta)$ stands for the left end point of the subinterval to which s belongs.

- Define "intermediate" approximating processes:

$$\hat{Y}^\varepsilon_t = y_0 + \frac{1}{\varepsilon}\int_0^t \tilde{g}(X^\varepsilon_{s(\delta)}, \hat{Y}^\varepsilon_s)\mathrm{d}s + \frac{1}{\sqrt{\varepsilon}}\int_0^t h(X^\varepsilon_{s(\delta)}, \hat{Y}^\varepsilon_s)\mathrm{d}^{\mathrm{I}} W_s,$$

and

$$\hat{X}^\varepsilon_t = x_0 + \int_0^t f(X^\varepsilon_{s(\delta)}, \hat{Y}^\varepsilon_s)\mathrm{d}s + \int_0^t \sigma(X^\varepsilon_s)\mathrm{d}B_s.$$

- To estimate $\|X^\varepsilon - \bar{X}\|_\infty$, we estimate

$$\|X^\varepsilon - \hat{X}^\varepsilon\|_\infty \quad \text{and} \quad \|\hat{X}^\varepsilon - \bar{X}\|_\infty, \quad \text{separately.}$$

Possible Future Developments

- First attempt in framework of RP theory.
- On the other hand, not truly deep. $\exists$ Room for sophistication even in the same formulation, e.g., $L^1 \rightsquigarrow L^p$? and $\| \cdot \|_\infty \rightsquigarrow \| \cdot \|_{\alpha-\mathrm{Hld}} \quad (\alpha < H)$?
- Weak AP in "fully-coupled" setting?
- 3rd level case (i.e., $\frac{1}{4} < H \leq \frac{1}{3}$) ?
- Other problems associated with AP for fast-slow systems: Normal deviation, FW-type large deviation, etc.

♠ Fast component Y^ε is driven by standard BM W to ensure ergodicity for the sol. $Y^{\xi,\phi}$ of "frozen SDE".

(Q) Can one replace W by FBM like this?

$$\begin{cases} \mathrm{d}X_t^\varepsilon = f\left(X_t^\varepsilon, Y_t^\varepsilon\right)\mathrm{d}t + \sigma\left(X_t^\varepsilon, Y_t^\varepsilon\right)\mathrm{d}B_t^{H_1}, \\ \mathrm{d}Y_t^\varepsilon = \dfrac{1}{\varepsilon} g\left(X_t^\varepsilon, Y_t^\varepsilon\right)\mathrm{d}t + \dfrac{1}{\varepsilon^{H_2}} h\left(X_t^\varepsilon, Y_t^\varepsilon\right)\mathrm{d}W_t^{H_2}. \end{cases}$$

Here, $(B_t^{H_1})$ and $(W_t^{H_2})$ are independent FBM's.

(cf) $\exists$ Very recent preprint by X.M.Li-Sieber in Young framework. $H_1, H_2 \in (1/2, 1)$, $h \equiv \mathrm{const.}$ $\rightsquigarrow$ weak AP. Hairer(-Ohashi)'s version of ergodicity for FBM is used.

The End