Functional central limit theorems for non-symmetric random walks on nilpotent covering graphs

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Pathwise Stochastic Analysis and Applications

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♠ **This talk is based on joint work with**
Satoshi ISHIWATA (Yamagata University),
Ryuya NAMBA (Ritsumeikan University).

- *Central limit theorems for non-symmetric random walks on nilpotent covering graphs: Part I, Electron. J. Probab. (2020), No. 86, pp. 1–46.*
- *Central limit theorems for non-symmetric random walks on nilpotent covering graphs: Part II, Potential Analysis (2021+), online first article, 40 pages.*

♣ **In this talk, we discuss CLTs for non-symmetric RWs on nilpotent covering graphs from a viewpoint of discrete geometric analysis developed by Toshikazu Sunada with Motoko Kotani.**

discrete geometric analysis **:**

harmonic analysis on infinite graphs with periodicity

(Yves Lejan's work on *Markov paths, loops and fields* **...)**

♣ **We study the "most natural realization" of the graph and capture (the natural lift of) a distorted Brownian rough path**

$$(B_{s,t}, \mathbb{B}_{s,t} + A_{s,t}) \quad \text{with} \quad A_{s,t} \in so(d)$$

through the CLT-scaling limit, simultaneously.

♠ **Related works :**

- **Breuillard–Friz–Huessman ('09): Brownian rough path**
- **Bayer–Friz ('13),**
- **Chevyrev ('18): Extension to Lévy processes, ... etc.**

- **Lejay–Lyons ('05): homogenization**
- **Friz–Oberhauser ('09),**
- **Friz–Gassiat–Lyons ('15): magnetic Brownian rough path**

- **Lopusanschi–Simon ('18), • Lopusanschi–Orenshtein ('18),**
- **Deuschel–Orenshtein–Perkowski ('19), ... etc.**

♣ **(Many) probabilist's approach:**

Realize the lattice into $\mathbb{R}^d$ or $\mathbb{G}^{(2)}(\mathbb{R}^d)$ firstly, then study limit theorems by using the given $\mathbb{R}^d$-coordinate.

♣ **Geometer's approach:**

Study the most "natural realization" of the lattice through limit theorems. $\Longrightarrow$

harmonic realization with the Albanese metric

- **"harmonic coordinate": Papanicolaou–Varadhan ('79), Kozolov ('85), ... etc**

Nilpotent Covering Graphs

♠ A locally finite connected graph $X = (V, E)$ is called a nilpotent covering graph if

there exists a finitely generated torsion free
nilpotent group Γ acting on X on the left freely, and

its quotient $X_0 = (V_0, E_0) := \Gamma \setminus X$ is a finite graph.

(In other words, X is a covering graph of a finite graph X_0 whose covering transformation group Γ is nilpotent.)

- torsion free: If $\gamma \neq 1_\Gamma$ and $\gamma^n = 1_\Gamma$, then $n = 0$.
- nilpotent: There exists some $r \in \mathbb{N}$ such that

$$\Gamma \supset [\Gamma, \Gamma] \supset \cdots \supset \Gamma^{(r)} (:= [\Gamma, \Gamma^{(r-1)}]) = \{1_\Gamma\}$$

- $\pi : X \to X_0$: covering map

♠ X is called a crystal lattice if Γ is abelian ($r = 1$).

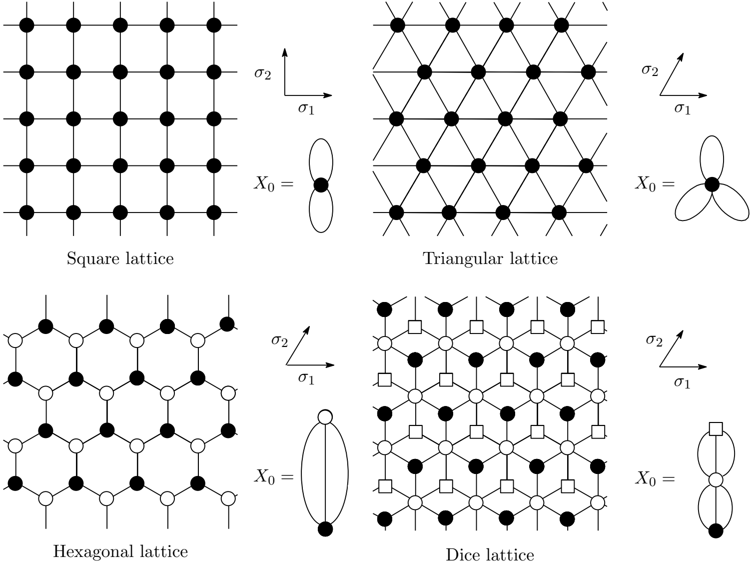


Figure : Crystal lattices with $\mathbf{\Gamma} = \langle \boldsymbol{\sigma_1}, \boldsymbol{\sigma_2} \rangle \cong \mathbb{Z}^2$

3D disctrete Heisenberg group : $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_1^{-1}, \gamma_2^{-1}, \gamma_3^{-1} \rangle$

$$\gamma_1\gamma_3 = \gamma_3\gamma_1, \ \gamma_2\gamma_3 = \gamma_3\gamma_2, \ [\gamma_1, \gamma_2](= \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}) = \gamma_3$$

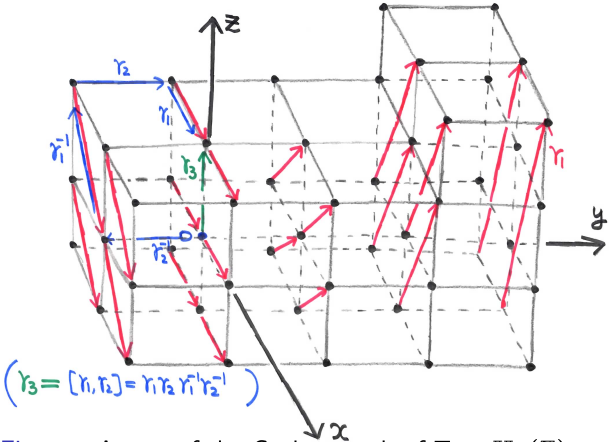


Figure : A part of the Cayley graph of $\Gamma = \mathbb{H}_3(\mathbb{Z})$

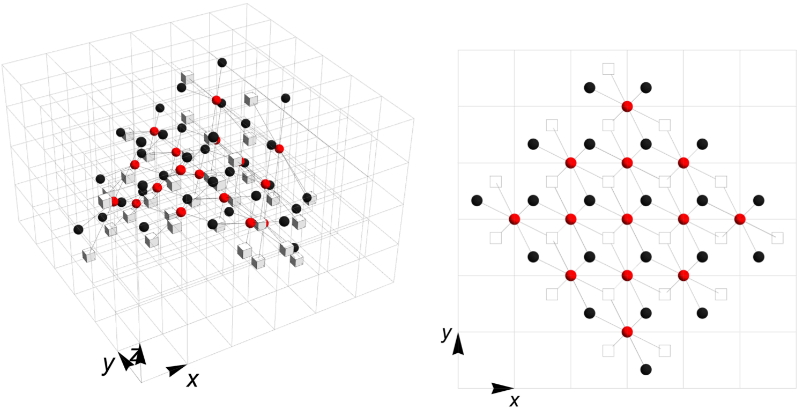


Figure : A part of 3D-Heisenberg dice lattice and the projection of it on xy-plane (by Shoichi Fujimori (Hiroshima University))

Random Walks on X

- For an edge $e \in E$, the origin, the terminus and the inverse edge of e are denoted by $o(e), t(e)$ and $\overline{e}$, respectively.
- $E_x := \{e \in E \,|\, o(e) = x\}$, $(x \in V)$.
- A RW on X is characterized by giving the one-step transition probability $p : E \longrightarrow (0,1)$ satisfying the Γ-invariance,

$$p(e) > 0 \quad (e \in E), \qquad \& \qquad \sum_{e \in E_x} p(e) = 1 \quad (x \in V).$$

$\Longrightarrow$ This induces a time homogeneous Markov chain

$$(\Omega_x(X), \mathbb{P}_x, \{w_n\}_{n=0}^{\infty}),$$

where $\Omega_x(X)$ stands for the set of all paths in X starting at x.

- By Γ-invariance of p, we may consider the RW on X_0; $(\Omega_{\pi(x)}(X_0), \mathbb{P}_{\pi(x)}, \{w_n\}_{n=0}^{\infty})$.

♠ $Lf(x) := \sum_{e \in E_x} p(e) f\big(t(e)\big)$: transition operator.

♠ n-step transition probability; $p(n,x,y) := L^n \delta_y(x)$.

♠ By the Perron-Frobenius theorem,

$$\exists!\, m : V_0 \longrightarrow (0,1] : L\text{-invariant measure, s.t.}$$

$$\sum_{x \in V_0} m(x) = 1 \quad \& \quad {}^t Lm(x) = m(x) \quad (x \in V_0).$$

♠ We also write $m : V \longrightarrow (0,1]$ for the lift of m to X, and introduce the conductance by

$$\widetilde{m}(e) := p(e) m\big(o(e)\big) \quad (e \in E)$$

♠ We define the homological direction γ_p of the RW by

$$\gamma_p := \sum_{e \in E_0} \widetilde{m}(e) e \in \mathrm{H}_1(X_0, \mathbb{R}).$$

♠ **RW: (m-)symmetric** $\overset{\text{def}}{\Longleftrightarrow}$ $\widetilde{m}(e) = \widetilde{m}(\overline{e})$ $\overset{\text{iff}}{\Longleftrightarrow}$ $\gamma_p = 0$.

Our Problem

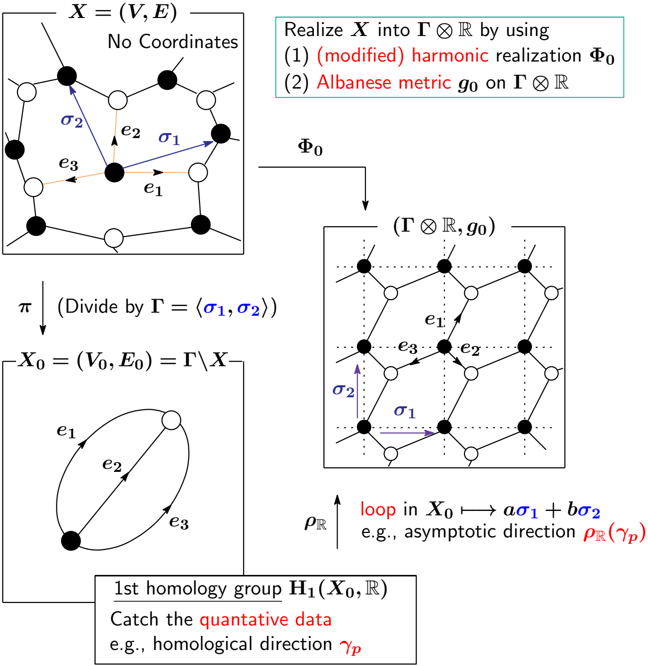
Functional CLT (Donsker's invariance principle)

♣ **Abelian case: Ishiwata–K–Kotani ('17)**

- *Long time asymptotics of non-symmetric random walks on crystal lattices, J. Funct. Anal. (2017), pp. 1553–1624.*

$$\left(\frac{\Phi_0(w_{[nt]}) - [nt]\rho_{\mathbb{R}}(\gamma_p)}{\sqrt{n}}\right)_{t\geq 0} \underset{n\to\infty}{\Longrightarrow} (B_t)_{t\geq 0} \quad \text{, where}$$

$\rho_{\mathbb{R}} : \mathrm{H}_1(X_0, \mathbb{R}) \twoheadrightarrow \Gamma \otimes \mathbb{R} (\cong \mathbb{Z}^d \otimes \mathbb{R} = \mathbb{R}^d)$,
$\Phi_0 : X \to (\Gamma \otimes \mathbb{R}, g_0)$ is the **"standard realization"**, and
(B_t): standard BM on $\Gamma \otimes \mathbb{R}$ with **Albanese metric g_0**.


$X = (V, E)$
No Coordinates
σ_2
e_2
σ_1
e_3
e_1
Realize X into $\Gamma \otimes \mathbb{R}$ by using
(1) (modified) harmonic realization Φ_0
(2) Albanese metric g_0 on $\Gamma \otimes \mathbb{R}$
Φ_0
$(\Gamma \otimes \mathbb{R}, g_0)$
e_1
e_3
e_2
σ_2
σ_1
π
(Divide by $\Gamma = \langle \sigma_1, \sigma_2 \rangle$)
$X_0 = (V_0, E_0) = \Gamma \backslash X$
e_1
e_2
e_3
$\rho_{\mathbb{R}}$
loop in $X_0 \longmapsto a\sigma_1 + b\sigma_2$
e.g., asymptotic direction $\rho_{\mathbb{R}}(\gamma_p)$
1st homology group $\mathbf{H}_1(X_0, \mathbb{R})$
Catch the quantative data
e.g., homological direction γ_p

Nilpotent Lie Group

♠ How to realize the Γ-nilpotent covering graph X into some continuous space ?

[Malćev ('51)]

$\exists G = G_\Gamma$: connected & simply connected nilpotent Lie group such that Γ is isomorphic to a cocompact lattice in $(G, \cdot)$.

♠ By a certain deformation of the product $\cdot$ on G, we may assume that G is a stratified Lie group of step r. Namely, its Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ satisfies

$$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}^{(i)}; \quad [\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \begin{cases} \subset \mathfrak{g}^{(i+j)} & (i+j \leq r), \\ = \{0_{\mathfrak{g}}\} & (i+j > r), \end{cases}$$

and $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(i)}]$ $(i = 1, \ldots, r-1)$.

Example: 3D Heisenberg group $\mathbb{H}^3(\mathbb{R})(= \mathbb{G}^{(2)}(\mathbb{R}^2))$

▷ $\Gamma = \mathbb{H}^3(\mathbb{Z}) := \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{Z} \right\} \left(\underset{\text{lattice}}{\hookrightarrow} G = \mathbb{H}^3(\mathbb{R}) \right).$

▷ $\mathfrak{g} = \mathrm{Lie}(G) = \left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$

▷ $X_1 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, X_2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, X_3 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

$$\left(\rightsquigarrow [X_1, X_2] = X_3, \ [X_1, X_3] = [X_2, X_3] = 0_{\mathfrak{g}}. \right)$$

▷ $G = \mathbb{H}^3(\mathbb{R})$: a **(free) nilpotent Lie group of step 2**, i.e.,

$$\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)}; \ \mathfrak{g}^{(1)} = \mathrm{span}_{\mathbb{R}}\{X_1, X_2\}, \ \mathfrak{g}^{(2)} := \mathrm{span}_{\mathbb{R}}\{X_3\}.$$

- If $\Gamma = \mathbb{G}^{(r)}(\mathbb{Z}^d)$, we may take $G = \mathbb{G}^{(r)}(\mathbb{R}^d)$ (r-step free nilpotent Lie group over $\mathbb{R}^d$). A lift of distorted Brownian rough path can be regarded as a $\mathbb{G}^{(r)}(\mathbb{R}^d)$-valued path.

- We identify G with $\mathbb{R}^n$ through the canonical coordinates of the 1st kind:

$$G \ni \exp\Big(\sum_{k=1}^{r} \underbrace{\sum_{i=1}^{d_k} x_i^{(k)} X_i^{(k)}}_{\in \mathfrak{g}^{(k)}}\Big)$$

$$\longleftrightarrow (x^{(1)}, x^{(2)}, \dots, x^{(r)}) \in \mathbb{R}^{d_1+d_2+\cdots+d_r},$$

where

- $\mathfrak{g} = (\mathfrak{g}^{(1)}, g_0) \oplus \mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(r)}$.
- $\mathfrak{g}^{(k)} = \mathrm{span}_{\mathbb{R}}\{X_1^{(k)}, \dots, X_{d_k}^{(k)}\}$ $(k = 1, \dots, r)$.
- $x^{(k)} = (x_1^{(k)}, \dots, x_{d_k}^{(k)}) \in \mathbb{R}^{d_k} \cong \mathfrak{g}^{(k)}$ $(k = 1, \dots, r)$.

Construction of the Albanese Metric on $\mathfrak{g}^{(1)}$

♠ **We induce a special flat metric on $\mathfrak{g}^{(1)}$, called the Albanese metric, by the following diagram:**

$$\begin{array}{ccc} (\mathfrak{g}^{(1)}, g_0) & \overset{\rho_{\mathbb{R}}}{\twoheadleftarrow} & \mathrm{H}_1(X_0,\mathbb{R}) \\ \updownarrow \text{dual} & & \updownarrow \text{dual} \\ \mathrm{Hom}(\mathfrak{g}^{(1)},\mathbb{R}) & \underset{{}^t\rho_{\mathbb{R}}}{\hookrightarrow} & \mathrm{H}^1(X_0,\mathbb{R}) \cong \big(\mathcal{H}^1(X_0), \langle\!\langle \cdot,\cdot \rangle\!\rangle_p\big). \end{array}$$

▷ **the set of modified-harmonic 1-forms :**

$$\mathcal{H}^1(X_0) := \Big\{\omega \in C^1(X_0,\mathbb{R}) \ : \sum_{e \in (E_0)_x} p(e)\omega(e) = \langle \gamma_p, \omega\rangle \Big\}$$

with $\langle\!\langle \omega, \eta \rangle\!\rangle_p := \sum_{e \in E_0} \widetilde{m}(e)\omega(e)\eta(e) - \langle \gamma_p, \omega\rangle\langle \gamma_p, \eta\rangle.$

Harmonic Realization of the Graph X into G

♠ We consider a Γ-equivariant map $\Phi : X = (V, E) \longrightarrow G$:

$$\Phi(\gamma x) = \gamma \cdot \Phi(x) \quad (\gamma \in \Gamma,\, x \in V).$$

Definition [Modified Harmonic Realization]

A realization $\Phi_0 : X \longrightarrow G$ is said to be **modified harmonic** if

$$\Delta\Big(\log \Phi_0\big|_{\mathfrak{g}^{(1)}}\Big)(x) = \rho_{\mathbb{R}}(\gamma_p) \quad (x \in V),$$

where $\Delta := L - I$: the discrete Laplacian on X.

▷ $\rho_{\mathbb{R}} : \mathrm{H}_1(X_0, \mathbb{R}) \twoheadrightarrow \mathfrak{g}^{(1)}$: the surjective linear map defined by

$$\rho_{\mathbb{R}}([c]) := \log(\sigma_c)\big|_{\mathfrak{g}^{(1)}} \quad \text{for } [c] \in \mathrm{H}_1(X_0, \mathbb{R})$$

s.t. $\sigma_c \in \Gamma(\hookrightarrow G)$ satisfies $\sigma_c \cdot o(\tilde{c}) = t(\tilde{c})$ on X.

▷ $\rho_{\mathbb{R}}(\gamma_p)$ is called the **($\mathfrak{g}^{(1)}$-)asymptotic direction** of the RW. We emphasize that

$$\gamma_p = 0 \quad \substack{\Longrightarrow \\ \not\Longleftarrow} \quad \rho_{\mathbb{R}}(\gamma_p) = 0_{\mathfrak{g}}.$$

♠ Such Φ_0 is uniquely determined up to $\mathfrak{g}^{(1)}$-translation, however, it has the **ambiguity** in $(\mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(r)})$-component!
As an example of realizations, we may consider the **Albanese map Φ_0** defined by

$$ {}_{\mathrm{Hom}(\mathfrak{g}^{(1)},\mathbb{R})}\Big\langle \omega, \log \Phi_0(x)\big|_{\mathfrak{g}^{(1)}} \Big\rangle_{\mathfrak{g}^{(1)}} = \int_{x_*}^{x} \widetilde{\omega} \qquad (x \in V),$$

where $\widetilde{\omega}$ is a lift of $\omega = {}^t\rho_{\mathbb{R}}(\omega) \in \mathrm{H}^1(X_0, \mathbb{R})$ to X and

$$\int_{x_*}^{x} \widetilde{\omega} = \int_c \widetilde{\omega} := \sum_{i=1}^{n} \widetilde{\omega}(e_i)$$

for a path $c = (e_1, \ldots, e_n)$ with $o(e_1) = x_*$ and $t(e_n) = x$.

The Geodesic Interpolation of the RW

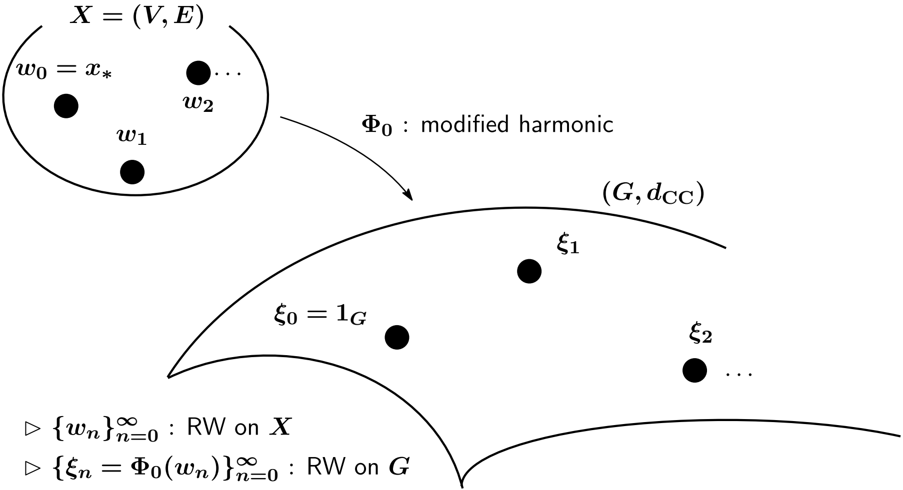
♠ **We introduce the 1-parameter group of dilations $\{\tau_\varepsilon\}_{\varepsilon \geq 0}$ on G:**

$$G \ni (x^{(1)}, x^{(2)}, \dots, x^{(r)}) \overset{\tau_\varepsilon}{\longmapsto} (\varepsilon x^{(1)}, \varepsilon^2 x^{(2)}, \dots, \varepsilon^r x^{(r)}) \in G.$$

♠ **We equip G with the Carnot-Carathéodory metric:**

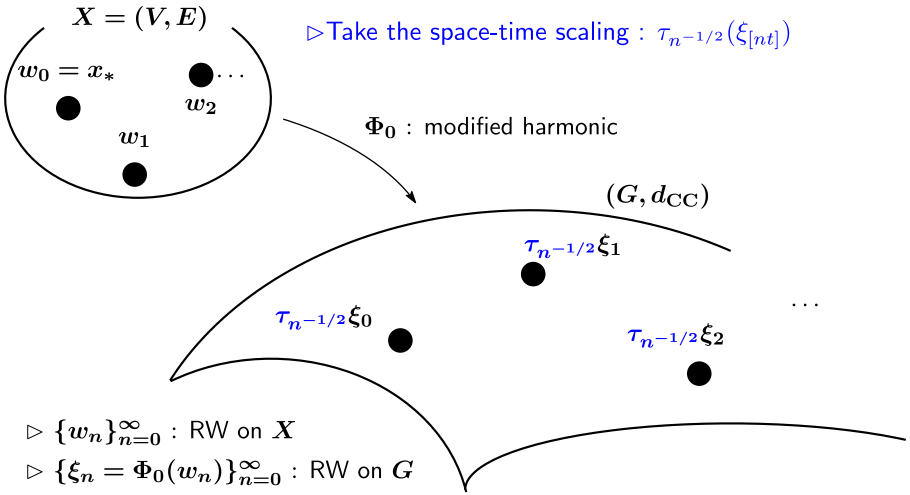
$$d_{\mathrm{CC}}(g,h) := \inf \Big\{ \int_0^1 \|\dot{c}(t)\|_{g_0} \, dt \, \Big| \, c \in \mathrm{AC}([0,1];G), \; c(0) = g, \, c(1) = h, \, \dot{c}(t) \in \mathfrak{g}^{(1)}_{c(t)} \Big\} \quad (g,h \in G).$$

♠ **(G, d_{CC}) is not only a metric space but also a geodesic space.**



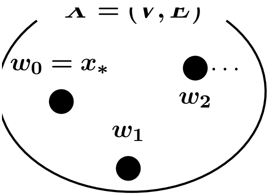
- ▷ $\{w_n\}_{n=0}^{\infty}$: RW on X
- ▷ $\{\xi_n = \Phi_0(w_n)\}_{n=0}^{\infty}$: RW on G

▷Take the space-time scaling : $\tau_{n^{-1/2}}(\xi_{[nt]})$



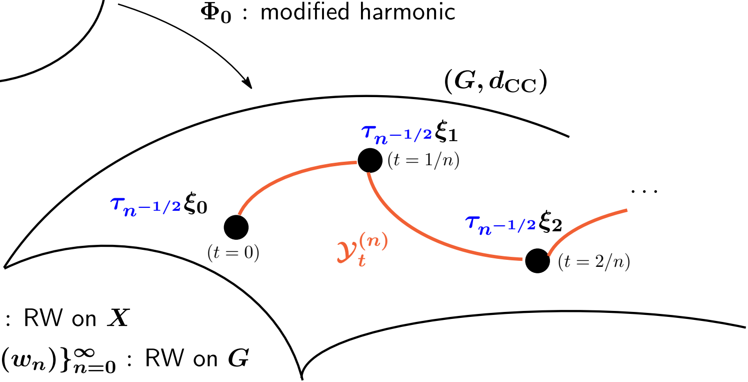
▷ $\{w_n\}_{n=0}^{\infty}$: RW on X

▷ $\{\xi_n = \Phi_0(w_n)\}_{n=0}^{\infty}$: RW on G



$\triangleright$ Take the space-time scaling : $\tau_{n^{-1/2}}(\xi_{[nt]})$

$\triangleright$ $\mathcal{Y}_t^{(n)}$: given by the d_{CC}-geodesic interpolation



$\triangleright$ $\{w_n\}_{n=0}^{\infty}$: RW on X

$\triangleright$ $\{\xi_n = \Phi_0(w_n)\}_{n=0}^{\infty}$: RW on G

The Main Result

♠ **We extend elements in $\mathfrak{g}$ to the left invariant C^∞-vector fields on G in the usual manner.**

Theorem (Ishiwata-K-Namba: Part I)

Under $\rho_{\mathbb{R}}(\gamma_p) = 0_{\mathfrak{g}}$, we have, for all $\alpha < 1/2$,

$$(\mathcal{Y}_t^{(n)})_{0 \le t \le 1} \underset{n\to\infty}{\Longrightarrow} (Y_t)_{0 \le t \le 1} \quad \text{in } \mathcal{C}_{1_G}^{\alpha\text{-Höl}}([0,1], G).$$

▷ $(Y_t)_{0 \le t \le 1}$ **: G-valued diffusion process which solves the SDE**

$$dY_t = \sum_{i=1}^{d} V_i(Y_t) \circ dB_t^i + \underbrace{\beta(\Phi_0)}_{\in \mathfrak{g}^{(2)}}(Y_t)\, dt, \quad Y_0 = 1_G.$$

▷ $\{V_1, \dots, V_d\}$ **: an ONB of $(\mathfrak{g}^{(1)}, g_0)$.**

▷ $(B_t^1, \dots, B_t^d)_{0 \le t \le 1}$ **: an $\mathbb{R}^d$-standard BM with $B_0^i = 0$.**

▷ generator : $\mathcal{A} = \frac{1}{2}\sum_{i=1}^{d} V_i^2 + \beta(\Phi_0)$ on $C_c^\infty(G)$.

($\dashrightarrow$ sub-Laplacian on G with $\mathfrak{g}^{(2)}$-drift)

▷ A special element in $\mathfrak{g}^{(2)}$:

$$\beta(\Phi_0) := \sum_{e \in E_0} \widetilde{m}(o(e)) \log\Big(\Phi_0(o(\widetilde{e}))^{-1} \cdot \Phi_0(t(\widetilde{e}))\Big)\Big|_{\mathfrak{g}^{(2)}},$$

where $\widetilde{e}$ stands for a lift of $e \in E_0$ to X.

Proposition

(1) RW : (m-)symmetric $\Longrightarrow \beta(\Phi_0) = 0_{\mathfrak{g}}$.

(2) $\beta(\Phi_0)$ is independent of the choice of $\mathfrak{g}^{(2)}$-component.

Important Remark

Replacing Φ_0 by a general periodic realization Φ in the definition of $\mathcal{Y}^{(n)}$, we have the same CLT. On the other hand, the information of the modified harmonic realization Φ_0 still remains in the drift term.

Some Comments

- **LLN for $\{\log \xi_n|_{\mathfrak{g}^{(1)}}\}_{n=0}^{\infty}$:**

$$\lim_{n\to\infty} \frac{1}{n} \log \xi_n(c)|_{\mathfrak{g}^{(1)}} = \rho_{\mathbb{R}}(\gamma_p), \quad \mathbb{P}_{x_*}\text{-a.s. } c \in \Omega_{x_*}(X).$$

- **By applying Yamato ('79), Kunita ('80), Ben Arous ('89), Castell ('93), etc, we can solve the SDE explicitly.**

$$Y_t = \exp\Big(t\beta(\Phi_0) + \sum_{i=1}^{d_1} B_t^i V_i$$

$$+ \sum_{0\le i<j\le d_1} \frac{1}{2} \int_0^t (B_s^i \circ dB_s^j - B_s^j \circ dB_s^i)[V_i, V_j] + \sum_{k=3}^{r} \sum_I c_t^I(B) V^I \Big)(1_G)$$

- **In the case $\Gamma = \mathbb{G}^{(r)}(\mathbb{Z}^{d_1}), G = \mathbb{G}^{(r)}(\mathbb{R}^{d_1})$, the limiting diffusion $(Y_t)_{0\le t\le 1}$ corresponds to the natural lift of the distorted Brownian rough path**

$$\overline{\mathbf{B}} = \Big(B_{s,t}, \mathbb{B}_{s,t} + (t-s)\overline{\beta}(\Phi_0) \Big)_{0\le s\le t\le 1}$$

Sketch of the Proof

♠ To obtain the main theorem, it is essential to prove

Lemma [tightness]

We assume $\rho_{\mathbb{R}}(\gamma_p) = 0_{\mathfrak{g}}$. Then the family of probability measures $\{\mathrm{P}^{(n)} := \mathbb{P}_{x_*} \circ (\mathcal{Y}^{(n)})^{-1}\}_{n=1}^{\infty}$ is **tight** in

$$\mathcal{C}^{\alpha\text{-Höl}}_{1_G}([0,1], G) := \overline{H^1_{1_G}([0,1]; G)}^{\|\cdot\|_{\alpha\text{-Höl}}},$$

where

- ▷ $H^1_{1_G}([0,1]; G)$: Cameron-Martin subspace in $C_{1_G}([0,1]; G)$.
- ▷ $\|x\|_{\alpha\text{-Höl}} := \sup_{0 \le s < t \le 1} \dfrac{d_{\mathrm{CC}}(x_s, x_t)}{(t-s)^{\alpha}} \quad (x \in \mathcal{C}^{\alpha\text{-Höl}}_{1_G}([0,1], G))$

♠ **How to prove tightness ?**
By induction for the step number $k = 1, \ldots, r$.

$(\mathcal{P}_k)$

Under $\rho_{\mathbb{R}}(\gamma_p) = 0_{\mathfrak{g}}$, the family

$$\left\{\mathrm{P}^{(n;k)} := \mathbb{P}_{x_*} \circ (\mathcal{Y}^{(n;k)})^{-1}\right\}_{n=1}^{\infty}$$

is tight in $\mathcal{C}^{\alpha\text{-Höl}}_{1_G}\left([0,1], \mathbb{R}^{d_1+\cdots+d_k}\right)$, where

- $(\mathcal{Y}^{(n;k)}_t)_{0\le t\le 1}$: $\mathbb{R}^{d_1+\cdots+d_k}$**-valued truncated stochastic process of** $(\mathcal{Y}^{(n)}_t = \mathcal{Y}^{(n;r)}_t)_{0\le t\le 1}$.

- **several martingale ineq's $\to$ $(\mathcal{P}_1)$ & $(\mathcal{P}_2)$ $\to$ Hölder regularity**
- **By employing an idea (of the proof) of Lyons' extension theorem in rough path theory, we can show $(\mathcal{P}_k)$ $(3 \le k \le r)$.**

Another Kind of CLT (Weakly Asymmetric Case)

- ♠ We introduce a family of transition probabilities $\{p_\varepsilon\}_{0\leq\varepsilon\leq 1}$ by $p_\varepsilon := p_0 + \varepsilon q$, where

$$p_0(e) := \frac{1}{2}\Big(p(e) + \frac{m(t(e))}{m(o(e))}p(\overline{e})\Big), \; q(e) := \frac{1}{2}\Big(p(e) - \frac{m(t(e))}{m(o(e))}p(\overline{e})\Big).$$

  It is the linear interpolation between p_0 and $p_1(= p)$.

  $\Longrightarrow \;\; \gamma_{p_\varepsilon} = \varepsilon\gamma_p, \; \rho_{\mathbb{R}}(\gamma_{p_\varepsilon}) = \varepsilon\rho_{\mathbb{R}}(\gamma_p).$

- ♠ $G = G_{(\varepsilon)}$: nilpotent Lie group whose Lie algebra is

$$(\mathfrak{g}^{(1)}, g_0^{(\varepsilon)}) \oplus (\oplus_{i=2}^r \mathfrak{g}^{(i)}).$$

- ♡ Continuity of the Albanese metric $g_0^{(\varepsilon)}$ w.r.t. ε.
- ♠ Take a (p_ε-)modified harmonic realization $\Phi_0^{(\varepsilon)} : X \to G$:

$$\Delta\Big(\log \Phi_0^{(\varepsilon)}\big|_{\mathfrak{g}^{(1)}}\Big)(x) = \varepsilon\rho_{\mathbb{R}}(\gamma_p) \quad (x \in V),$$

♠ For $0 \le \varepsilon \le 1$, we define a G-valued stochastic process $(\mathcal{Y}^{(\varepsilon,n)})_{0\le t\le 1}$ given by "(d_{CC}-)geodesic interpolation" of $\left(\tau_{n^{-1/2}}(\Phi_0^{(\varepsilon)}(w_{[nt]}))\right)_{0\le t\le 1}$.

Theorem (Ishiwata-K-Namba: Part II)

Under additional natural two assumptions on $\{\Phi_0^{(\varepsilon)}\}_{\varepsilon>0}$, we have, for all $\alpha < 1/2$,

$$(\mathcal{Y}_t^{(n^{-1/2},n)})_{0\le t\le 1} \underset{n\to\infty}{\Longrightarrow} (\widetilde{Y}_t)_{0\le t\le 1} \quad \text{in } \mathcal{C}_{1_G}^{\alpha\text{-Höl}}([0,1], G_{(0)}).$$

▷ $(\widetilde{Y}_t)_{0\le t\le 1}$: $G_{(0)}$-valued diffusion process which solves the SDE

$$d\widetilde{Y}_t = \sum_{i=1}^{d} V_i^{(0)}(\widetilde{Y}_t) \circ dB_t^i + \underbrace{\rho_{\mathbb{R}}(\Phi_0^{(0)})}_{\in \mathfrak{g}^{(1)}}(\widetilde{Y}_t)\, dt, \quad \widetilde{Y}_0 = 1_G.$$

▷ $\{V_1^{(0)}, \ldots, V_d^{(0)}\}$: an ONB of $(\mathfrak{g}^{(1)}, g_0^{(0)})$.

▷ $(B_t^1, \ldots, B_t^d)_{0\le t\le 1}$: an $\mathbb{R}^d$-standard BM with $B_0^i = 0$.

Final Remarks

♣ The non-centered case $\rho_{\mathbb{R}}(\gamma_p) \neq 0_{\mathfrak{g}}$:

Applying the idea of Girsanov transform to nilpotent settings, we can generalize the CLT to the non-centered case.

♠ **We define** $F : V_0 \times \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R}) \longrightarrow (0, \infty)$ **by**

$$F_x(\lambda) := \sum_{e \in (E_0)_x} p(e) \exp\Big({}_{\mathrm{Hom}(\mathfrak{g}^{(1)},\mathbb{R})} \big\langle \lambda, \log d\Phi_0(\widetilde{e})\big|_{\mathfrak{g}^{(1)}} \big\rangle_{\mathfrak{g}^{(1)}} \Big) \qquad \big(x \in V_0,\ \lambda \in \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R})\big),$$

where $\widetilde{e}$ **denotes a lift of** $e \in E_0$ **to** X **and**

$$d\Phi_0(e) := \Phi_0\big(o(e)\big)^{-1} \cdot \Phi_0\big(t(e)\big) \qquad (e \in E).$$

♠ **Let** $\mathfrak{p} : E_0 \longrightarrow \mathbb{R}$ **be a function defined by**

$$\mathfrak{p}(e) := p(e) \exp\Big(\big\langle \lambda_*\big(o(e)\big), \log d\Phi_0(\widetilde{e})|_{\mathfrak{g}^{(1)}} \big\rangle \Big) F_{o(e)}^{-1}\big(\lambda_*(o(e)\big),$$

where $\lambda_*(x)$ **is the minimizer of** $F_x : \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R}) \to (0, \infty)$.

♠ $\rho_{\mathbb{R}}(\gamma_{\mathfrak{p}}) = 0_{\mathfrak{g}}$, Φ_0 is $\mathfrak{p}$-harmonic, and centered $\Longrightarrow p = \mathfrak{p}$.

Theorem (Ishiwata–K–Namba: Part I)

We have, for all $\alpha < 1/2$,

$$(\mathcal{Y}^{(n,\mathfrak{p})})_{n=1}^{\infty} \underset{n\to\infty}{\Longrightarrow} \widehat{Y} \quad \text{in } \mathcal{C}_{1_G}^{\alpha\text{-Höl}}\big([0,1], G_{(0)}\big).$$

▷ $\widehat{Y} = (\widehat{Y}_t)_{0\le t\le 1}$: G-valued diffusion process which solves

$$d\widehat{Y}_t = \sum_{i=1}^{d_1} V_i^{(\mathfrak{p})}(\widehat{Y}_t) \circ dB_t^i + \beta_{(\mathfrak{p})}(\Phi_0)(\widehat{Y}_t)\, dt, \quad \widehat{Y}_0 = 1_G.$$

▷ $\{V_1^{(\mathfrak{p})}, V_2^{(\mathfrak{p})}, \dots, V_{d_1}^{(\mathfrak{p})}\}$: an ONB of $(\mathfrak{g}^{(1)}, g_0^{(\mathfrak{p})})$.

▷ $(B_t^1, B_t^2, \dots, B_t^{d_1})_{0\le t\le 1}$: an $\mathbb{R}^{d_1}$-standard BM with $B_0^i = 0$.

▷

$$\beta_{(\mathfrak{p})}(\Phi_0) := \sum_{e\in E_0} \mathfrak{p}(e)\mathfrak{m}\big(o(e)\big) \log\Big(\Phi_0\big(o(\widetilde{e})\big)^{-1}\cdot\Phi_0\big(t(\widetilde{e})\big)\Big)\Big|_{\mathfrak{g}^{(2)}}.$$

♠ **In Breuillard's expository article ('06), difficulties of CLT under non-centered setting is mentioned.**
(After Raugi ('78)'s old work, there are few papers.)

♣ **Speed of convergence:**

$$\sup_{x,y\in V}\Big|p(n,x,y)m(y)^{-1} - K\frac{|G/\Gamma|}{m(X_0)}\mathcal{H}\big(n,\Phi_0(x),\Phi_0(y)\big)\Big| \le Cn^{-\frac{D+1}{2}},$$

where $K := \gcd\{n \in \mathbb{N} \mid p(n,x,x) > 0\}$ **(period of the RW),**
D **is the (polynomial) volume growth rate of** Γ**.**

♠ **Related works : • Alexopoulos (around '00), • Ishiwata ('04), • Breuillard ('05), • Diaconis–Hough ('18), • Hough ('19), • Namba (arXiv: 2011.13783)**

The End

Thank you for your attention

Merci de nous accorder votre attention