On transport type rough fluid equations

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March 11th, 2021
CIRM: Pathwise Stochastic Analysis and Applications

Introduction and overview

- Rough path theory offers an attractive framework to model the effects of computationally unresolvable fluctuations on the resolvable parts of fluid flows. [Palmer et al., 2009].
- In [Leahy et al., 2020] we formulated geometric fluid dynamics on rough diffeomorphisms and characterized solutions of rough fluid PDEs as critical points of constrained action functionals.
- In [Leahy et al., 2021], we establish local well-posedness and a blow-up criterion for perfect incompressible fluids on geometric rough paths within the framework of unbounded rough drivers [Bailleul and Gubinelli, 2017].

Hamilton's principle: The Euler-Lagrange equations

- Let Q be a manifold and $L \in C^1(TQ, \mathbb{R})$.
- For given $q_1, q_2 \in Q$ and an interval $[t_1, t_2]$, define

$$C(q_1, q_2, [t_1, t_2]) = \left\{ q \in C^2([t_1, t_2]; Q) : q(t_1) = q_1, \quad q(t_2) = q_2 \right\}.$$

Theorem

A curve $q \in C(q_1, q_2, [t_1, t_2])$ is a critical point of

$$S(q) = \int_{t_1}^{t_2} L(q(t), \dot{q}(t)) \, \mathrm{d}t$$

if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \right] = \frac{\partial L}{\partial q}(q, \dot{q}).$$

- Let (q, v, p) be the local coordinates for the bundle $E := TQ \oplus T^*Q$.
- $C_E(q_1, q_2, [t_1, t_2]) :=$
  $\left\{(q, v, p) \in C^1([t_1, t_2]; E) : q(t_1) = q_1, \quad q(t_2) = q_2\right\}.$

Theorem ([Yoshimura and Marsden, 2006])

A curve $(q, v, p) \in C_E(q_1, q_2, [t_1, t_2])$ is a critical point of

$$S(q, v, p) = \int_{t_1}^{t_2} \left(L(q(t), u(t)) + p(t) \cdot (\dot{q}(t) - v(t))\right) \, \mathrm{d}t$$

if and only if

$$\dot{q} = v, \quad \dot{p} = \frac{\partial L}{\partial q}, \quad p = \frac{\partial L}{\partial v} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

Lie group setting

- Let $Q = G$ be a Lie group with Lie algebra $\mathfrak{g} = T_e G \cong \mathfrak{X}_R(G)$.
- Assume that the Lagrangian $L \in C^1(TG; \mathbb{R})$ is right-invariant so that

$$\int_{t_1}^{t_2} L(q(t), \dot{q}(t))\,\mathrm{d} = \int_{t_1}^{t_2} \ell(u(t))\,\mathrm{d}t, \quad u := TR_{q^{-1}}\dot{q} = \dot{q}q^{-1} : [t_1, t_2] \to \mathfrak{g},$$

  where $\ell : \mathfrak{g} \to \mathbb{R}$ is defined by $\ell(u) := L(e, u),\ u \in \mathfrak{g}$.
- Assume the functional derivative $\frac{\delta \ell}{\delta u} : \mathfrak{g} \to \mathfrak{g}^*$ defined by

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \ell(u + \epsilon\delta u) = \langle \frac{\delta \ell}{\delta u}(u), \delta u \rangle_{\mathfrak{g}}, \quad \forall\, \delta u \in \mathfrak{g}.$$

  is a diffeomorphism.
- Recall that $\mathrm{ad} = T_e\,\mathrm{Ad} = T_e TL_g \circ TR_{g^{-1}} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is given by

$$\mathrm{ad}_\xi\, u = -[u, \xi], \quad \forall \xi, u \in \mathfrak{g}.$$

- Let $\mathrm{ad}^* : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}^*)$ denote its dual relative to the pairing $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$.

Euler-Poincaré reduction

- Let (q, u, λ) be the local coordinates for the bundle $E := G \times \mathfrak{g} \oplus \mathfrak{g}^*$.
- $C_E(q_1, q_2, [t_1, t_2]) :=$
  $\left\{(q, u, \lambda) \in C^1([t_1, t_2]; E) : q(t_1) = q_1, \quad q(t_2) = q_2\right\}.$

Theorem ([Yoshimura and Marsden, 2007])

The following are equivalent for a curve $q \in C(q_1, q_2, [t_1, t_2])$:

- *q is a critical point of $S(q) = \int_{t_1}^{t_2} L(q(t), \dot{q}(t))\,\mathrm{d}t$.*
- *q satisfies the Euler-Lagrange equations.*
- *$(q, u = \dot{q}q^{-1}, \lambda = \frac{\delta \ell}{\delta u}) \in C_E(q_1, q_2, [t_1, t_2])$ is a critical point of*

$$S(q, u, \lambda) = \int_{t_1}^{t_2} \ell(u(t)) + \langle \lambda(t), \dot{q}(t)q^{-1}(t) - u(t)\rangle_{\mathfrak{g}}.$$

- *$u = \dot{q}q^{-1} \in \mathfrak{g}$ satisfies the Euler-Poincaré equations [Holm et al., 1998]*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta \ell}{\delta u} + \mathrm{ad}_u^* \frac{\delta \ell}{\delta u} = 0.$$

Topological ideal hydrodynamics [Arnold, 1966]

- Let $G = \mathrm{Diff}^s_{\mu_g}(M)$, $s > \frac{d}{2} + 1$, be the topological group of Sobolev volume-preserving diffeomorphisms on (M, g).
- $\mathfrak{g} := T_e G \cong \mathfrak{X}^s_{\mu_g}(M)$ is isomorphic to the space of divergence-free Sobolev vector fields.
- Define for $q \in C(q_1, q_2, [0, T])$ with $u = \dot{q} q^{-1} : [0, T] \to \mathfrak{X}^s_{\mu_g}(M)$:

$$S(q) = \int_0^T \int_M g_{q_t}\left(\dot{q}_t, \dot{q}_t\right) \mu_g \, \mathrm{d}t = \int_0^T \int_M |u|^2 \mu_g \, \mathrm{d}t.$$

- There exists a smooth geodesic spray [Ebin and Marsden, 1970]:

$$\nabla_{\dot{q}} \dot{q} = -\nabla p \circ q, \qquad \text{Euler-Lagrange equations,}$$

$$\Leftrightarrow \quad \partial_t u + \nabla_u u = -\nabla p, \quad \text{Euler-Poincare equations,}$$

where ∇ is the Levi-Civita connection on $\mathrm{Diff}^s(M)$ induced by the 'weak' energy metric.

Co-adjoint operator on diffeomorphism group

- Until further specified, all quantities are smooth.
- Let $G = \mathrm{Diff}(M)$ and $\mathfrak{g} = \mathfrak{X}(M) = \mathfrak{X}$.
- The canonical dual is given by $\mathfrak{g}^* = \mathfrak{X}^\vee = \Omega^1 \otimes \Omega^d$ with pairing

$$\langle \alpha \otimes \mu, u \rangle_{\mathfrak{X}} = \int_M \alpha(u)\mu, \quad \alpha \otimes \mu \in \mathfrak{X}^\vee, \;\; u \in \mathfrak{X}.$$

- Similarly, we can take volume-preserving $G = \mathrm{Diff}_{\mu_g}$ and $\mathfrak{g} = \mathfrak{X}_{\mu_g}$ with the corresponding $\mathfrak{X}^\vee_{\mu_g}$ using the Hodge decomposition.
- We have $\mathrm{ad}_u\, v = -[u, v] = -\pounds_u v$ for all $u, v \in \mathfrak{X}$.
- Integrating by parts, we find [Holm et al., 1998]

$$\langle \alpha \otimes \mu, \mathrm{ad}_u\, v \rangle_{\mathfrak{X}} = \langle \pounds_u(\alpha \otimes \mu), v \rangle_{\mathfrak{X}},$$

  and thus the co-adjoint operator is the Lie derivative

$$\mathrm{ad}^*_u = \pounds_u : \mathfrak{g}^* \to \mathrm{Der}(\mathfrak{g}^*).$$

Advected quantities and the Lagrangian

- Let $\mathfrak{A}$ be a summand of tensor field bundles.
- Paths in $\mathfrak{A}$ are advected quantities such as temperature and density.
- Denote the corresponding duality pairing by $\langle \cdot, \cdot \rangle_{\mathfrak{A}} : \mathfrak{A}^\vee \times \mathfrak{A} \to \mathbb{R}$.
- Define the momentum map $\diamond : \mathfrak{A}^\vee \times \mathfrak{A} \to \mathfrak{X}^\vee$ [Holm et al., 1998]:

$$\langle b, \pounds_u a \rangle_{\mathfrak{A}} = -\langle b \diamond a, u \rangle_{\mathfrak{X}} \quad \forall a \in \mathfrak{A},\ b \in \mathfrak{A}^\vee,\ u \in \mathfrak{X}.$$

- Let $\ell : \mathfrak{X} \times \mathfrak{A} \to \mathbb{R}$ denote the fluid Lagrangian, which is physically determined.
- Assume that $\frac{\delta \ell}{\delta u} : \mathfrak{X} \times \mathfrak{A} \to \mathfrak{X}^\vee$ and $\frac{\delta \ell}{\delta a} : \mathfrak{X} \times \mathfrak{A} \to \mathfrak{A}^\vee$ are continuous:

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \ell(u + \epsilon \delta u, a + \epsilon \delta a) = \langle \frac{\delta \ell}{\delta u}(u, a), \delta u \rangle_{\mathfrak{X}} + \langle \frac{\delta \ell}{\delta a}(u, a), \delta a \rangle_{\mathfrak{A}}$$

Geometric rough flows

- Let $K \in \mathbb{N}$, $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, and $\mathbf{Z} \in \mathscr{C}_g^\alpha(\mathbb{R}_+; \mathbb{R}^K)$ be an α truly rough geometric rough path.
- Let $\xi = (\xi_k)_{k=1}^K \in \mathfrak{X}^K$ be a collection of smooth vector fields.
- Let $\mathrm{Diff}_{\mathbf{Z}}$ denote the space of rough flows $\{\eta_t\} \subset \mathrm{Diff}$:

$$d\eta_t = v_t \circ \eta_t \mathrm{d}t + \sigma_t \circ \eta_t \mathrm{d}\mathbf{Z}_t, \quad \eta_0 = \mathrm{id},$$

  for arbitrarily given $(v, \sigma) \in C_T^\alpha(\mathfrak{X}) \times C_T^\infty(\mathfrak{X}^K)$.
- For a given controlled rough path $\lambda \in \mathscr{D}_{Z,T}(\mathfrak{X}^\vee)$, define

$$\int_0^T \langle \lambda_t, \mathrm{d}\eta_t \eta_t^{-1} \rangle_{\mathfrak{x}} := \int_0^T \langle \lambda_t, v_t \rangle_{\mathfrak{x}} \, \mathrm{d}t + \int_0^T \langle \lambda_t, \sigma_t \rangle_{\mathfrak{x}} \, \mathrm{d}\mathbf{Z}_t.$$

H-P variational principle on geometric rough paths

Theorem

A curve (η, u, λ) *is a critical point of*

$$S^{HP_{\mathbf{Z}}}(\eta, u, \lambda) = \int_0^T \ell(u_t, \eta_{t*}a_0)\mathrm{d}t + \langle \lambda_t, d\eta_t\eta_t^{-1} - u_t\mathrm{d}t - \xi\mathrm{d}\mathbf{Z}_t\rangle_{\mathfrak{X}}.$$

iff $(\eta, u, \lambda = \frac{\delta\ell}{\delta u}) \in \mathrm{Diff}_{\mathbf{Z}} \times C_T^{\alpha}(\mathfrak{X}) \times \mathfrak{D}_{Z,T}(\mathfrak{X}^{\vee})$ *satisfy*

$$\begin{aligned}
\mathrm{d}\eta_t &= u_t \circ \eta_t \mathrm{d}t + \xi \circ \eta_t \mathrm{d}\mathbf{Z}_t, \\
\mathrm{d}\frac{\delta\ell}{\delta u} + \pounds_{u_t}\frac{\delta\ell}{\delta u}\mathrm{d}t + \pounds_{\xi}\frac{\delta\ell}{\delta u}\mathrm{d}\mathbf{Z}_t &\overset{\mathfrak{X}^{\vee}}{=} \frac{\delta\ell}{\delta a} \diamond a_t \mathrm{d}t.
\end{aligned}$$

By the rough Lie chain rule, $a_t = \eta_{t*}a_0$ *satisfies*

$$\mathrm{d}a + \pounds_{u_t}a\mathrm{d}t + \pounds_{\xi}a\mathrm{d}\mathbf{Z}_t \overset{\mathfrak{A}}{=} 0.$$

Kelvin circulation theorem

The density $D \in C_T^{\alpha}(\Omega^d)$ is an advected quantity:

$$\mathrm{d}D + \pounds_u D \mathrm{d}t + \pounds_\xi D \mathrm{d}\mathbf{Z}_t = 0 \quad \Leftrightarrow \quad D_t = \eta_{t*} D_0.$$

Theorem

Let Γ denote a compactly embedded one-dimensional smooth submanifold of M. If D_0 is non-vanishing, then

$$\int_{\eta_t \Gamma} \frac{1}{D_t} \frac{\delta \ell}{\delta u}(u_t, a_t) = \int_{\Gamma} \frac{1}{D_0} \frac{\delta \ell}{\delta u}(u_0, a_0) + \int_0^t \int_{\eta_s \Gamma} \frac{1}{D_s} \frac{\delta \ell}{\delta a}(u_s, a_s) \diamond a_s \mathrm{d}s.$$

Rough incompressible Euler system

- Let $\dot{\mathfrak{X}}_{\mu_g}$ denote the incompressible and 'harmonic-free' vector fields.
- Define the kinetic energy Lagrangian $\ell : \dot{\mathfrak{X}}_{\mu_g} \to \mathbb{R}$ by

$$\ell(u) = \frac{1}{2} \int_M g(u, u)\mu_g.$$

- Thus, $\frac{\delta \ell}{\delta u} = [u^\flat \otimes \mu_g] \in \mathfrak{X}^\vee_{\mu_g}$, which is an equivalence class.
- The corresponding equation for a critical point is given by:

$$\begin{cases} \mathrm{d}u^\flat + \pounds_u u \mathrm{d}t + \pounds_\xi u^\flat \mathrm{d}\mathbf{Z}_t = -\mathbf{d}(\mathrm{d}q_t - 2^{-1}|u|^2 \mathrm{d}t) - \mathrm{d}h^\flat_t \\ \mathrm{div}_{\mu_g} u^\flat = 0, \quad H(u^\flat) = 0, \quad H(q) = 0, \\ u|_{t=0} = u_0, \end{cases}$$

where $\mathbf{d}$ denotes the exterior derivative, and the pressure term q and harmonic term h correspond to two constraints.

Vorticity formulation

Since the exterior derivative $\mathbf{d}$ commutes with Lie derivatives, $\mathbf{d}^2 = 0$, and $\mathbf{d}h^\flat = 0$, we get that the vorticity $\omega = \mathbf{d}u^\flat \in \Omega_2$ is 'advected'

$$\begin{cases} \mathrm{d}\omega + \pounds_u \omega \mathrm{d}t + \pounds_\xi \omega \mathrm{d}\mathbf{Z}_t = 0, \\ u = \sharp \mathbf{d}_g^\star \Delta^{-1} \omega =: \mathrm{BS}(\omega), \\ \omega|_{t=0} = \mathbf{d}u_0^\flat. \end{cases}$$

In particular, the dynamics of the vorticity preserve exactness and thus there are no Lagrange multipliers, which is exploited in our proof of well-posedness.

Rough incompressible fluid on the torus

- Let $K \in \mathbb{N}$, $p \in [2,3)]$, and $\mathbf{Z} = (Z, \mathbb{Z}) \in \mathscr{C}_g^{p-\mathrm{var}}(\mathbb{R}_+, \mathbb{R}^K)$
- Let $d \in \{2, 3, \ldots\}$, $m \geq \lfloor \frac{d}{2} \rfloor + 2$, and $\xi \in W^{m+2,\infty}(\mathbb{T}^d, \mathbb{R}^{d\times K})$.
- For $\nu \geq 0$, consider a system of rough PDEs on $\mathbb{T}^d \times [0, T]$ given by

$$\begin{cases} \mathrm{d}u + u \cdot \nabla u \mathrm{d}t + (\xi_k \cdot \nabla u + (\nabla \xi_k)u)\mathrm{d}Z_t^k = \nu \Delta u - \nabla \mathrm{d}q_t - \mathrm{d}h_t, \\ \operatorname{div} u = 0, \quad \int_{\mathbb{T}^d} u \, dV = 0, \quad \int_{\mathbb{T}^d} q \, dV = 0, \\ u|_{t=0} = u_0, \quad q|_{t=0} = 0, \quad h|_{t=0} = 0 \end{cases} \tag{1}$$

  where $u : [0, T] \times \mathbb{T}^d \to \mathbb{R}^d$ is a divergence and mean-free vector field, $q : [0, T] \times \mathbb{T}^d \to \mathbb{R}$ is a mean-free time-integrated pressure, and $h : [0, T] \to \mathbb{R}^d$ is a time-integrated harmonic constant.
- The case $\nu > 0$ was studied in [Leahy et al., 2020]. The $\nu = 0$ (i.e., Euler) is a new result.
- Henceforth, denote $L_k\phi = \xi_k \cdot \nabla\phi + (\nabla\xi_k)\phi$.

- Let $\dot{P}$ denote the projection onto mean and divergence-free vector fields. Recall that $\dot{P} + Q + H = \mathrm{id}$, where Q and H denote the gradient and harmonic projection, respectively.
- Let $\mathbf{Z}^n = (Z^n, \mathbb{Z}^n)$ be the canonical lift of a sequence of piecewise smooth paths converging to $\mathbf{Z}$ in the rough path topology.
- For a given $u_0 \in W^{m,2}_\sigma$, there exists a maximal solution $u^n \in C([0, T_{\max}); \dot{W}^{m,2}_\sigma)$ to the system (1) with $\mathbf{Z}$ replaced by $\mathbf{Z}^n$.
- It follows that (c.f., [Bailleul and Gubinelli, 2017])

$$
\begin{aligned}
\delta u^n_{st} + \int_s^t \dot{P}[u^n_r \cdot \nabla u^n_r - \nu \Delta u^n_r] dr &= - \int_s^t \dot{P} L_k u^n_r \mathrm{d} Z^{n,k}_r \\
&= -\dot{P} L_k u^n_s Z^{n,k}_{st} + \dot{P} L_k \dot{P} L_l u_s \mathbb{Z}^{n,lk}_{st} + u^{n,P,\sharp}_{st},
\end{aligned}
$$

where $u^{n,P,\sharp}_{st} \in C_2^{\frac{p}{3}-\mathrm{var}}([0,T]; \dot{W}^{m-3,2}_\sigma)$.

Definition of solution of velocity equation

Definition

A path

$$u \in L_T^\infty \dot{W}_\sigma^{m,2} \cap C_T \dot{W}_\sigma^{m-3,2}$$

is said to be a $W^{m,2}$- solution of (1) on $[0,T]$ if

$$u_{st}^{\dot{P},\natural} := \delta u_{st} + \int_s^t \dot{P}[u_r \cdot \nabla u_r - \nu \Delta u_r]\, dr + \dot{P} L_k u_s Z_{st}^k - \dot{P} L_k \dot{P} L_l u_s \mathbb{Z}_{st}^{lk}$$

satisfies $u^{\dot{P},\natural} \in C_{2,T,\mathrm{loc}}^{\frac{p}{3}-\mathrm{var}} \dot{W}_\sigma^{m-3,2}$. We say u is a $W^{m,2}$-solution of (1) on $[0,T)$ if u is a solution on the interval $[0, T-\epsilon]$ for all $\epsilon > 0$.

Reconstruction of pressure and harmonic constant

Proposition

If u is a $W^{m,2}$-solution of (1) on $[0,T]$, then there exists unique paths $q \in C_T^{p-\mathrm{var}}\dot{W}^{m-2,2}$ and $h \in C_T^{p-\mathrm{var}}\mathbb{R}^d$ initiating from zero such that

$$u^{\natural}_{st} := \delta u_{st} + \int_s^t (u_r \cdot \nabla u_r - \nu \Delta u_r)\, dr + L_k u_s Z^k_{st} - L_k L_l u_s \mathbb{Z}^{lk}_{st} + \nabla \delta q_{st} + \delta h_{st}$$

satisfies $u^{\natural} \in C_{2,T,\mathrm{loc}}^{\frac{p}{3}-\mathrm{var}} W^{m-3,2}$.

We construct q and h via the sewing lemma:

$$\nabla q_t := -\int_0^t Q u_r \cdot \nabla u_r \, dr - \int_0^t Q L_k u_r \mathrm{d}Z^k_r$$

$$h_t := -\int_0^t \int_{\mathbb{T}^d} (\nabla \xi_k) u_t \, dV \mathrm{d}Z^k_r.$$

Equivalent vorticity formulation

Let $\dot{W}^{2,m}_{\mathbf{d}}$ denote the L^2-Sobolev space of m-times weakly differentiable functions taking values in the space of anti-symmetric matrices that are mean-free and have vanishing exterior derivative.

Proposition

If u is a $W^{m,2}$-solution of (1) on the interval $[0,T]$, then

$$\Omega = \operatorname{curl} u = (\nabla - D)u \in L^\infty_T \dot{W}^{m-1,2}_{\mathbf{d}} \cap C_T \dot{W}^{m-4,2}_{\mathbf{d}}$$

$$\Omega^\natural = \operatorname{curl} u^{\dot{P},\natural} \in C^{\frac{p}{3}-\mathrm{var}}_{2,T,\mathrm{loc}} \dot{W}^{m-4,2}_{\mathbf{d}}$$

satisfy with $\mathbf{L}_v \Phi = v \cdot \nabla \Phi + (\nabla v)\Phi + \Phi(Dv)$,

$$\Omega^\natural_{st} = \delta \omega_{st} + \int_s^t (\mathbf{L}_{u_r} \Omega_r - \nu \Delta \Omega_r)\, dr + \mathbf{L}_{\xi_k} \Omega_s Z^k_{st} - \mathbf{L}_{\xi_k} \mathbf{L}_{\xi_l} \Omega_s \mathbb{Z}^{lk}_{st}, \qquad (2)$$

Conversely, if there exists ω and $\omega^\natural$ such that (2) holds with $\omega_0 = \operatorname{curl} u_0$ and $u := \mathrm{BS}\,\omega$, then u is a $W^{m,2}$- solution of (1).

Existence and uniqueness

Theorem

There exists a constant $C = C(d, m, p, |\xi|_{W^{m+2,\infty}})$ *such that for any time* T_* *satisfying*

$$\exp\left(C(1 + \omega_{\mathbf{Z}}(0, T_*))\right) T_* < \frac{1}{1 + |u_0|_{W^{m,2}}},$$

there exists a unique $W^{m,2}$*-solution*

$$u \in C_{w,T_*} \dot{W}^{m,2}_{\sigma} \cap C_{T_*} \dot{W}^{m-,2}_{\sigma} \cap C^{p-\mathrm{var}}_{T_*} \dot{W}^{m-1,2}_{\sigma}.$$

of (1) *on the interval* $[0, T_*]$*. If* $\xi \in W^{m+4,\infty}$*, then* $u \in C_{T_*} \dot{W}^{m,2}_{\sigma}$.

Theorem

The above solution can be uniquely extended to a maximal time interval

$$u \in C_w([0, T_{\max}); \dot{W}^{m,2}_{\sigma}) \cap C([0, T_{\max}); \dot{W}^{m-,2}_{\sigma}) \cap C^{p-\mathrm{var}}([0, T_{\max}); \dot{W}^{m-1,2}_{\sigma}).$$

Moreover, if $T_{\max} < \infty$*, then* $\limsup_{t \uparrow \max} |u_t|_{W^{m,2}} = \infty$.

Continuous dependence on data

Theorem

Assume that $\{(u_0^n, \nu^n, \xi^n, \mathbf{Z}^n\}) \in \dot{W}_\sigma^{m,2} \times [0,\infty) \times W^{m+2,\infty} \times C_g^{p-\mathrm{var}}$ *converges to* $\{(u_0, \nu, \xi, \mathbf{Z}\})$.

Let $\{(u^n, T_{\max}^n)\}_{n=1}^\infty$ *and* $(u, T_{\max}))$ *denote the maximal solutions corresponding to the data* $\{(u_0^n, \nu^n, \xi^n, \mathbf{Z}^n\}$ *and* $(u_0, \nu, \xi, \mathbf{Z})$*, respectively.*

Then there exists an $N \in \mathbb{N}$ *such that for all* $n \geq N$, $T_{\max}^n > T_{\max}$. *Moreover,* $\{u^n\}_{n=N}^\infty$ *converges to* u *in* $C([0, T_{\max}); \dot{W}_\sigma^{m-,2})$ *and the weak-star topology of* $L^\infty([0, T_{\max}); \dot{W}_\sigma^{m,2})$.

In particular, if $\nu^n \to 0$, then the $W^{m,2}$-Navier-Stokes solution u^{ν^n} solutions tend to the Euler solution u.

Theorem

Let u denote the unique maximal $W^{m,2}$-solution of (1). There exists a positive constants $C_1 = C_1(d, m)$ and $C_2 = C_2(p, d, m, |\xi|_{W^{m+2,\infty}})$ such that for all $t \in [0, T_{\max})$,

$$|u_t|_{W^{m,2}} \leq C_1(1 + |u_0|_{W^{m,2}}) \exp\left(C_2 \varpi_{\mathbf{Z}}(0, t) \exp\left(C_2 \int_0^t |\Omega_r|_{L^\infty} \, \mathrm{d}r\right)\right).$$

Moreover, if $T_{\max} < \infty$, then $\int_0^{T_{\max}} |\Omega_t|_{L^\infty} \, \mathrm{d}t = \infty$.

Corollary

Let u denote the unique maximal $W^{m,2}$-solution of (1) and $\Omega = \operatorname{curl} u$. Then for all $t \in [0, T_{\max})$,

$$|\Omega_t|_{L^\infty} = |\Omega_0|_{L^\infty}, \text{ if } \nu = 0, \quad |\Omega_t|_{L^\infty} \le |\Omega_0|_{L^\infty}, \text{ if } \nu > 0. \tag{3}$$

Thus, by the BKM blow-up criterion, there exists a unique $W^{m,2}$-solution of (1) on $[0, \infty)$. Moreover, there exists positive constants $C_1 = C_1(m)$ and $C_2 = C_2(p, m, |\xi|_{W^{m+2,\infty}})$ such that for all $t \in \mathbb{R}_+$,

$$|u_t|_{W^{m,2}} \le C_1(1 + |u_0|_{W^{m,2}}) \exp\left(C_2 \varpi_{\mathbf{Z}}(0,t) \exp\left(C_2 t |\Omega_t|_{L^\infty}\right)\right).$$

Various results such as Wong-Zakai approximation, Large deviation principle, and the existence of a random dynamical system are consequences.

Open issues

- Obtain solution estimates using only the velocity formulation. The issue we face is the incompressibility constraint. In particular, changing the structure of the noise in the velocity in a zero-order way leads to difficulties due to the changed structure in the vorticity equation.
- Uniqueness in 2D for $\omega_0 \in L^\infty$. Existence seems possible, but we don't have enough regularity at the level of vorticity to take the L^2-norm of the difference of two solutions. In the deterministic case, one proves uniqueness via the velocity formulation [Yudovich, 1963, Majda et al., 2002]. Perhaps one can use a rough flow approach like in the stochastic setting [Brzeźniak et al., 2016].

Future outlook for applications

- Develop numerical schemes for rough PDEs
- Calibrate ξ for a cross-validated set of Gaussian rough paths (e.g. FBM with H) on a coarse-grid from direct numerical simulations of the underlying unperturbed fluid PDE.
- How can we update the parameters of the subgrid model with real observational data?
- DNS data ought to be good for initializing parameters and learning subgrid parameters. In the case of Lorenz 96, we can compare with rigorous homogenization results.

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Unbounded rough drivers

- Let $(E_n)_{0\le n\le 3}$ be a scale of Banach spaces possessing a smoothing $(J^\eta)_{\eta\in(0,1)}$. Denote $E_{-n} = E_n^*$.
- Assume $L_k \in \mathscr{L}(E_{n+1}, E_n)$, $n \in \{0,2\}$, $L_k L_l \in \mathscr{L}(E_{n+2}, E_n)$, $n \in \{0,1\}$.
- Assume that $\mu : [0,T] \to E_{-1}$ satisfies $|\delta\mu_{st}|_{E_{-1}} \le \omega_\mu(s,t)$.
- Assume that $f \in C_T E_{-0}$ is such that for all $\phi \in E_3$,

  $$\langle f^\natural_{st}, \phi\rangle := \langle \delta f_{st}, \phi\rangle + \langle \delta\mu_{st}, \phi\rangle + \langle f_s, (L_k^*\phi Z^k_{st} + L_l^* L_k^* \mathbb{Z}^{lk}_{st})\phi\rangle$$

  satisfies $f^\natural \in C^{\frac{p}{3}-\mathrm{var}}_{2,\omega_{\mathbf{Z}},T,\mathrm{loc}} E_{-3}$.
- There exist an $L = L(p) > 0$ such that for all $(s,t) \in \Delta_{[t_0,T]}$ with $\omega_{\mathbf{A}}(s,t) + \omega_\mu(s,t) \le L$,

  $$|f^\natural|^{\frac{p}{3}}_{\frac{p}{3}-\mathrm{var},[s,t],E_{-3}} \le C\left(\sup_{s\le r\le t} |f_r|_{-0}\,\omega_{\mathbf{Z}}(s,t)^{\frac{3}{p}} + \omega_\mu(s,t)\,\omega_{\mathbf{Z}}(s,t)^{\frac{1}{p}}\right),$$

  $$|f|^p_{p-\mathrm{var},[s,t],E_{-0}} \le C\left(\omega_\mu(s,t) + \sup_{s\le r\le t} |f_r|_{-0}(\omega_\mu(s,t)^{\frac{1}{p}} + \omega_{\mathbf{Z}}(s,t)^{\frac{1}{p}})\right).$$

- We form the system of equations of the derivatives of ω up to order $m-1$, $(\omega^{(m-1)}, \omega^{(m-1),\natural})$, and obtain a priori estimates of $|\omega^{\natural}|_{\frac{p}{3}-\mathrm{var}, W^{m-4,2}}$ and $|\omega|_{p-\mathrm{var}, W^{m-2,2}}$ in terms of $\sup_{s\leq r\leq t} |\omega|_{W^{m-1,2}}$ using URD estimates.
- We form the system of equation for $(\omega^{(m-1)} \otimes \omega^{(m-1)}, \omega^{(m-1),\otimes 2,\natural})$ and obtain a bound on $\omega^{(m-1),\otimes 2,\natural}$ using URD in the L^∞-scale. We then apply the rough Gronwall lemma [Deya et al., 2019] and Bihari's inequality to obtain estimates of $\sup_{s\leq r\leq t} |\omega|_{W^{m-1,2}}$, thereby closing the a priori estimates.
- To prove uniqueness, we work with the vorticity formulation and develop the equation for the square.