The structure group revisited

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Perspective on driven ODEs

Initial value problem for ODE with (rough) driver:
$\frac{du}{dt} = a(u)\xi, \quad u(0) = 0.$

Butcher/Rough-path point of view, all nonlinearities at once:
$u = u[a](t)$ (Gubinelli'10).

Inhomogeneous initial data, i. e. $\tilde{u}(0) = u_0$,
recovered via u-shift:
$\tilde{u} = u[a(\cdot + u_0)] + u_0$,
$\rightsquigarrow$ parameterization of solution manifold.

Re-centering, i. e. $u_1(1) = 0$,
recovered via suitable variable u-shift $\pi = \pi[a]$:
$u_1[a] = u[a(\cdot + \pi[a])] + \pi[a]$.

Perspective on driven PDEs

PDE with (rough) driver, e. g. gPAM:

$$(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})u = a(u)\xi \quad \text{mod polynomials.}$$

Parameterize solution manifold
by jets/space-time polynomials $p = p(x)$:

$u(x) = u[a,p](x)$ (Bruned&Chandra&Chevyrev&Hairer'19).

In this work: $u(x) = u[a,p](x)$ mod constants,
and p mod constants.

Guided by quasi-linear class:

$$(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})u = a(u)\frac{\partial^2 u}{\partial x_1^2} + \xi \quad \text{mod polynomials.}$$

Two actions on (a,p)-space

Set of $(a,p) \in \mathbb{R}[u] \times \mathbb{R}[x_1,x_2]/\mathbb{R}$.

$\mathbb{R}[x_1,x_2]/\mathbb{R} \cong \{\, p \in \mathbb{R}[x_1,x_2] \mid p(0)=0 \,\}$.

Action of $\mathbb{R}^2 \ni y$ by **shift**: $\Big(a\big(\cdot + p(y)\big)\,,\, p(\cdot + y) - p(y)\Big)$.

Action of $\mathbb{R}[x_1,x_2] \ni q$ by **tilt**: $\Big(a\big(\cdot + q(0)\big)\,,\, p + q - q(0)\Big)$.

This work is on the **representation** of these actions.

Ignore algebra structure of $\mathbb{R}[u] \ni a$ (cf. Faà-di Bruno),

and vector space structure of $\mathbb{R}[x_1,x_2]/\mathbb{R} \ni p$ (but affine).

Lifting, variable tilt, and monoid structure

Lift maps $(a,p) \mapsto \Big(a\big(\cdot + p(y)\big)\,,\, p(\cdot + y) - p(y)\Big)$,

$$(a,p) \mapsto \Big(a\big(\cdot + q(0)\big)\,,\, p + q - q(0)\Big)$$

to endomorphisms of algebra of functions π on (a,p)-space: $(\Gamma_y^*\pi)[a,p] = \pi\Big[a\big(\cdot + p(y)\big)\,,\, p(\cdot + y) - p(y)\Big]$,

$$(\Gamma_q^*\pi)[a,p] = \pi\Big[a\big(\cdot + q(0)\big)\,,\, p + q - q(0)\Big].$$

Extend to variable tilt $\{\pi^{(n)} = \pi^{(n)}[a,p]\}_n$:

$$(\Gamma^*\pi)[a,p] = \pi\Big[a\big(\cdot + \pi^{(0)}[a,p]\big)\,,\, p + \textstyle\sum_{n\neq 0} \pi^{(n)}[a,p]x^n\Big].$$

Monoid: If $\left\{\begin{array}{c} \pi^{(n)} \mapsto \Gamma^* \\ \pi'^{(n)} \mapsto \Gamma'^* \end{array}\right\}$ then $\pi^{(n)} + \Gamma^*\pi'^{(n)} \mapsto \Gamma^*\Gamma'^*$

(cf. Bruned&Chevyrev&Friz&Preiss'19)

– need inverse for re-centering.

Algebra of coordinates on (a,p)-space

Introduce coordinates on (a,p)-space
(arbitrarily fixing origin in (u,x)-space):

$$z_k[a] := \frac{1}{k!}\frac{d^k a}{du^k}(0),\ \ k\in\mathbb{N}_0, \qquad z_n[p] = \frac{1}{n!}\frac{d^n p}{dx^n}(0),\ \ n\in\mathbb{N}_0^2-\{(0,0)\}.$$

Seek T* (= (algebraic) dual of abstract model space T)
as linear subspace of $\mathbb{R}[[z_k, z_n]]$ (= formal power series algebra).

Formally get model Π from applying partial derivative

$\Pi_{k\geq 0}(\frac{\partial}{\partial z_k})^{\beta(k)}\,\Pi_{n\neq 0}(\frac{\partial}{\partial z_n})^{\beta(n)}$ for multi-index β,

to general solution $u = u[a,p]$.

Relation to the model

From multi-index β to equation for Π_β:

Set $\Pi_{e_n} = x^n$, for quasi-linear class solve inductively

$(\frac{\partial}{\partial x_2}-\frac{\partial^2}{\partial x_1^2})\Pi_0 = \xi$, $(\frac{\partial}{\partial x_2}-\frac{\partial^2}{\partial x_1^2})\Pi_{e_0} = \frac{\partial^2}{\partial x_1^2}\Pi_0$, $(\frac{\partial}{\partial x_2}-\frac{\partial^2}{\partial x_1^2})\Pi_{e_1} = \Pi_0\frac{\partial^2}{\partial x_1^2}\Pi_0, \ldots$

Use algebra $\mathbb{R}[[z_k, z_n]]$ to write more compactly:

for quasi-linear: $(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})\Pi = \sum_{k\geq 0} z_k \Pi^k \frac{\partial_1^2}{\partial x_1^2}\Pi + \xi 1$;

for gPAM: $(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})\Pi = \sum_{k\geq 0} z_k \Pi^k \xi$;

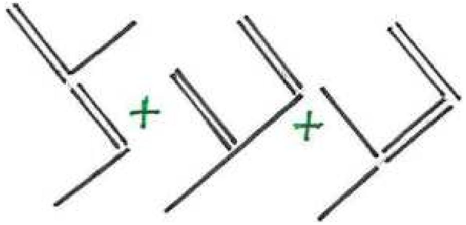
for ϕ^4: $(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})\Pi = \sum_{k\geq 0} z_k \Pi^k + \xi 1$.

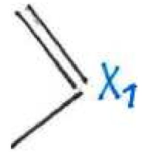
Dictionary to trees depends on class of equations

Translate monomials z^β into
linear combinations of trees (decorated edges and nodes).

For quasi-linear class:

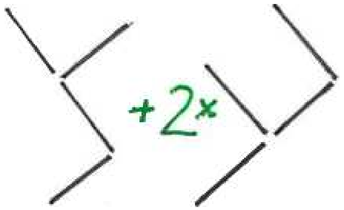
z_0 

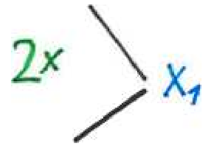
$z_0 z_1$ 

$z_1 z_{(1,0)}$ 

For gPAM-class / for ϕ^4-class:

$z_0 z_1$ / z_1 

$z_0^2 z_1 z_2$ / $z_1 z_2$ 

$z_0 z_2 z_{(1,0)}$ / $z_2 z_{(1,0)}$ 

Choice of abstract model space on dual level $\mathsf{T}^* \subset \mathbb{R}[[z_k z_n]]$
guided by population of model Π for class under consideration:
for gPAM: $\sum_{k\geq 0}(k\text{-}1)\beta(k) - \sum_{n\neq 0}\beta(n) = -1$; for ϕ^4: $\beta(0) = 0$.

Abstract model space T^* for quasi-linear class

Component Π_β populated iff $\beta \in \{e_n\}_{n\neq 0} \cup \{\beta |\, [\beta] \geq 0\,\}$,

where $[\beta] := \sum_{k\geq 0} k\beta(k) - \sum_{n\neq 0} \beta(n)$;

note $[\beta] = (\text{homogeneity in } u) - (\text{homogeneity in } p)$.

Yields natural decomposition $\mathsf{T}^* = \bar{\mathsf{T}}^* \oplus \tilde{\mathsf{T}}^*$.

Hairer's abstract model space $\widehat{=}\ \tilde{\mathsf{T}} \oplus \mathbb{R} \oplus \mathsf{T}$,

integration map $\mathcal{I}\ \widehat{=}\ \tilde{\mathsf{T}} \hookrightarrow \mathsf{T}$.

Towards a Lie algebra $\mathsf{L} \subset \mathrm{Der}(\mathbb{R}[[z_k, z_n]])$

Generator of u-shift: $D^{(0)} := \sum_{k\geq 0}(k+1)z_{k+1}\frac{\partial}{\partial z_k}$.

Generator of x^n-tilt: $D^{(n)} := \frac{\partial}{\partial z_n}$, $n \neq 0$.

Generator of x_1-shift: $\partial_1 := \sum_n (n_1+1) z_{n+(1,0)} D^{(n)}$,

(same for ∂_2).

Well-defined **derivations** on algebra $\mathbb{R}[[z_k, z_n]]$,
i. e. as elements of $\mathrm{Der}(\mathbb{R}[[z_k, z_n]])$.

Despite modding out constants,
commutators behave canonically:
$[D^{(n)}, D^{(m)}] = 0$, $[\partial_1, \partial_2] = 0$, $[D^{(n)}, \partial_1] = n_1 D^{(n-(1,0))}$
(same for ∂_2).

A pre-Lie structure on L ⊂ End(T*)

Extension to variable tilt: $z^{\gamma} D^{(n)} \in \mathsf{Der}(\mathbb{R}[[z_k, z_n]])$.
All $D \in \{\partial_i\}_{i=1,2} \cup \{z^{\gamma} D^{(n)}\}_{[\gamma]\geq 0,n}$ preserve T* and $\tilde{\mathsf{T}}^*$.

Note $\mathsf{Der}(\mathbb{R}[[z_k, z_n]]) \cong$ {vector fields on flat $\mathbb{R}[[z_k, z_n]]$}.
Hence Lie algebra product $[\cdot, \cdot]$
arises from a pre-Lie algebra product $\triangleleft$.

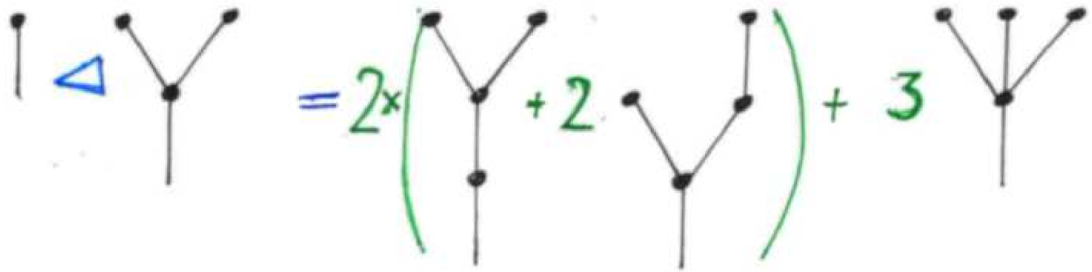
For our derivations: $\partial_1 \triangleleft D = 0$,
$D \triangleleft z^{\gamma} D^{(n)} = (D z^{\gamma}) D^{(n)}$, $\quad z^{\gamma} D^{(n)} \triangleleft \partial_1 = n_1 z^{\gamma} D^{(n-(1,0))}$.

No obvious relation between $\triangleleft$ and tree grafting

Grafting as a pre-Lie structure on (decorated) trees (BCCH'19, Bailleul&Bruned'21).

Have $z_0 D^{(0)} \triangleleft z_0^2 z_2 D^{(0)} = 2 z_0^2 z_1 z_2 D^{(0)} + 3 z_0^3 z_3 D^{(0)}$.

Interpretation for gPAM, according to our dictionary:



Gradations on pre-Lie algebra and definition of L

Two **gradations** on $\{\partial_i\} \cup \{z^\gamma D^{(n)}\}$ compatible with $\triangleleft$:

for $z^\gamma D^{(n)}$: $(1+[\gamma], \sum_m |m|\gamma(m) - |n|)$,

for ∂_1: $(0, |(1,0)|)$, for ∂_2: $(0, |(0,1)|)$.

Fix parameter $\alpha > 0$, introduce **homogeneity**

$\langle\gamma\rangle := \alpha(1+[\gamma]) + \sum_{m\neq 0} |m|\gamma(m)$.

Define $\mathsf{L} := \mathrm{span}\Big(\{\partial_i\}_{i=1,2} \cup \{z^\gamma D^{(n)}\}_{[\gamma]\geq 0, |n|<\langle\gamma\rangle}\Big)$, then

pre-Lie product $\triangleleft$ preserves L,

for $D \in \mathsf{L}$, $D^\dagger$ is strictly triangular w. r. t. $\langle\cdot\rangle$.

From L to Hopf algebra U(L)

Closedness under $\triangleleft$ yields Lie algebra L via

$[D, D'] = D \triangleleft D' - D' \triangleleft D.$

View L as direct sum indexed by

$\{(\gamma, n)\}_{[\gamma]\geq 0, |n|<\langle\gamma\rangle} \cup \{1, 2\}.$

Consider universal enveloping algebra

U(L) $:=$ T(L)$/[\cdot, \cdot]=$ Hopf algebra.

Lift representation ρ: U(L) $\to$ End(T*)

however no longer faithful (i. e. one-to-one).

A canonical basis of U(L)

Pre-Lie structure provides U(L) $\cong$ S(L) (Oudom&Guin'08).

Recall L $=$ direct sum indexed by $\{(\gamma, n)\}_{[\gamma]\geq 0, |n|<\langle\gamma\rangle} \cup \{1, 2\}$.

Hence S(L)

$=$ direct sum indexed by multi-indices J on $\{(\gamma, n)\}_{\ldots} \cup \{1, 2\}$.

Get canonical pairing between U(L) and

T$^+$ $:=$ direct sum indexed by J's.

Have canonical pairing between T* and

T $:=$ direct sum indexed by γ's.

Injection $\gamma \mapsto J = e_{(\gamma,n)}$ yields $\mathcal{J}_n$: T $\to$ T$^+$;

canonical definition of $x^m \in$ T$^+$.

Definition of Δ^+ and Δ by duality/pairing

Finiteness properties & gradedness imply:

(concatenation) product on U(L) yields

co-product $\Delta^+\colon \mathsf{T}^+ \to \mathsf{T}^+ \otimes \mathsf{T}^+$,

representation $\rho\colon \mathsf{U(L)} \to \mathsf{End}(\mathsf{T}^*)$ yields

co-module $\Delta\colon \mathsf{T} \to \mathsf{T}^+ \otimes \mathsf{T}$.

Get intertwining axioms of Hairer'15:

$$\Delta^+\mathcal{J}_n = (1 \otimes \mathcal{J}_n)\Delta + \sum_m \mathcal{J}_{m+n} \otimes \frac{x^m}{m!},$$

$$\Delta^+ x_1 = x_1 \otimes \{\} + \{\} \otimes x_1, \quad \text{same for } x_2.$$

Consistency of Δ with tree pruning, 4 examples

1) $\Delta z_0 z_1 = \{\} \otimes z_0 z_1 + \{z_0\} \otimes z_0 + 2\{1\} \otimes z_0^2.$

Interpretation in terms of trees for quasi-linearized class:

2) $\Delta z_2 z_x = \{\} \otimes z_2 z_x + 2\{x_1\} \otimes z_1 + 2\{1\} \otimes z_1 z_x$

$+ 2\{1, x_1\} \otimes z_0 + \{1, 1\} \otimes z_0 z_x$ where $z_x := z_{(1,0)}.$

Interpretation in terms of trees for quasi-linear class:

Reminiscent of Connes-Moscovici-Δ inside Butcher-Δ (Connes&Kreimer, see Brouder'04)

Consistency of Δ with tree pruning, 4 examples

3) $\Delta z_0^2 z_2 = \{\} \otimes z_0^2 z_2 + 2(\{z_0^2\} \otimes z_1 + \{z_0\} \otimes z_0 z_1 + \{1\} \otimes z_0^2 z_1)$
$+ 2\{1, z_0^2\} \otimes z_0 + \{z_0, z_0\} \otimes z_0 + 4\{1, z_0\} \otimes z_0^2 + 3\{1,1\} \otimes z_0^3.$

For gPAM reduces to

$$\Delta z_0^2 z_2 = \{\} \otimes z_0^2 z_1 + 2\{z_0\} \otimes z_0 z_1 + \{z_0, z_0\} \otimes z_0,$$

4) $\Delta z_1 z_2 = \{\} \otimes z_1 z_2 + \{z_2\} \otimes z_0 + \{1\} \otimes z_0 z_2 + 2\{z_1\} \otimes z_1$
$+ 4\{1\} \otimes z_1^2 + 3\{1,1\} \otimes z_0 z_1 + 2\{1, z_1\} \otimes z_0 + 2\{1,1,1\} \otimes z_0^2.$

For ϕ^4 reduces to

$$\Delta z_1 z_2 = \{\} \otimes z_1 z_2 + 2\{z_1\} \otimes z_1 + 4\{1\} \otimes z_1^2,$$

Definition of $\mathsf{G} \subset \mathsf{End}(\mathsf{T})$

Define $\mathsf{G} \cong \mathsf{Alg}(\mathsf{T}^+, \mathbb{R}) \; = \; \{\, f \in (\mathsf{T}^+)^* \mid f \text{ multiplicative} \,\}$.

Co-product $\Delta^+ \colon \mathsf{T}^+ \to \mathsf{T}^+ \otimes \mathsf{T}^+$ yields

product on G via $f \circ g = (f \otimes g)\Delta^+$.

co-module $\Delta \colon \mathsf{T} \to \mathsf{T}^+ \otimes \mathsf{T}$ yields

representation $\mathsf{G} \subset \mathsf{End}(\mathsf{T})$ via $\Gamma_f = (f \otimes 1)\Delta$.

Form of counter-term for quasi-linear class

Seek $(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})u + h = a(u)\frac{\partial^2}{\partial x_1^2} + \xi$ with

h local, i. e. independent of p, only on $u(x)$,

h deterministic, thus not explicitly dependent on x,

h covariant, i. e. $h[a](u+v) = h[a(\cdot + v)](u)$.

Parameterization by deterministic $c \in \mathsf{T}^*$,

with $D^{(n)}c = 0$ for $n \neq 0$, such that

$$h[a](u) = c[a(\cdot + u)] = \sum_\beta c_\beta \prod_{k\geq 0}(\frac{1}{k!}\frac{d^k a}{du^k}(u))^{\beta(k)}$$
$$= \sum_\beta c_\beta(a(u)) \prod_{k\geq 1}(\frac{1}{k!}\frac{d^k a}{du^k}(u))^{\beta(k)}.$$

Requires truncation, $\langle\cdot\rangle$ not coercive on e_0.

Structure group G compatible with renormalization

On level of stationary model Π

(rather polynomial $\otimes$ stationary)

$$(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})\Pi = \sum_{k\geq 0} z_k \Pi^k \frac{\partial^2}{\partial x_1^2}\Pi - \sum_{k\geq 0} \frac{1}{k!}\Pi^k D^{(0)}c + \xi 1.$$

Since $D^{(n)}c = 0$ for $n \neq 0$,

on the level of centered model Π_x,

related via $\Pi_x = F_x\Pi + \pi_x^{(0)}$ with $\{\tau_x^{(n)}\}_n \mapsto F_x \in \mathsf{G}$

$$(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})\Pi_x = \sum_{k\geq 0} z_k \Pi_x^k \frac{\partial^2}{\partial x_1^2}\Pi_x - \sum_{k\geq 0} \frac{1}{k!}\Pi_x^k D^{(0)}c + \xi 1.$$

BPHZ renormalization

On level of (r. h. s. of) stationary model,

there is a unique deterministic c,

with $c_\beta = 0$ unless $\beta(n) = 0$ for all $n \neq 0$, such that

$$\Pi^- := \sum_{k\geq 0} z_k \Pi^k \frac{\partial^2}{\partial x_1^2}\Pi - \sum_{k\geq 0} \frac{1}{k!}\Pi^k D^{(0)} c + \xi 1$$

satisfies $\mathbb{E}\Pi^-_\beta = 0$, provided $\beta(n) = 0$ for all $n \neq 0$.

If ξ invariant under $x_1 \rightsquigarrow -x_1$, then

also $\mathbb{E}\Pi^-_{\beta+e_{(1,0)}} = 0$ provided $\beta(n) = 0$ for all $n \neq 0$.

Annealed stochastic estimates on model (Π_x, Γ_{xy})

For (regularized) white noise, BPHZ-renormalization, and with $\alpha = \frac{1}{2}-$

$$\mathbb{E}^{\frac{1}{p}}|\Pi^{-}_{x\beta\lambda}(x)|^p \lesssim \lambda^{\langle\beta\rangle-2},$$

$$\mathbb{E}^{\frac{1}{p}}|\Pi_{x\beta}(y)|^p \lesssim |x-y|^{\langle\beta\rangle},$$

$$\mathbb{E}^{\frac{1}{p}}|\Gamma^{\gamma}_{xy\beta}|^p \lesssim |x-y|^{\langle\beta\rangle-\langle\gamma\rangle}.$$

in progress with P. Linares, M. Tempelmayr, P. Tsatsoulis

– without passing via trees.