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Càdlàg rough differential equations with reflecting boundary

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Reflected (rough) differential equations

Aim: We look for (Y, K) solving

$$Y_t = y + \int_0^t f_1(Y_s)\,\mathrm{d}A_s + \int_0^t f_2(Y_s)\,\mathrm{d}X_s + K_t, \quad t \in [0, T],$$

such that, for every $i = 1, \dots, n$,

(a) $Y_t^i \geq L_t^i$ for all $t \in [0, T]$,

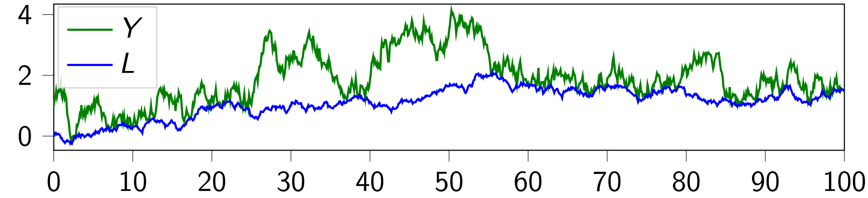
(b) $K^i \colon [0, T] \to \mathbb{R}$ is a non-decreasing function such that $K_0^i = 0$, and

$$\int_0^t (Y_s^i - L_s^i)\,\mathrm{d}K_s^i = 0, \quad t \in [0, T].$$

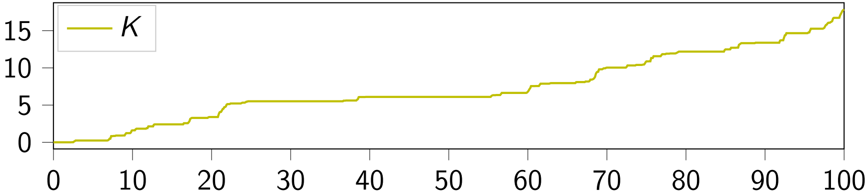
Input:

- $L \in D^p([0, T]; \mathbb{R}^d)$ ($\sim$ càdlàg of finite p-variation)
- $A \in D^q([0, T]; \mathbb{R}^d)$ for $q \in [1, 2)$, $X \in D^p([0, T]; \mathbb{R}^d)$ for $p \in (2, 3)$

Example plot of the solution Y and the barrier L:



Example plot of the reflector K:



Background - probabilistic setting

History on reflected SDEs:

- domains: Skorokhod '61, McKean '63, El Karoui '75, Tanaka '79, Lions & Sznitman '83, Saisho '87, ...
- time-dependent boundaries L: Skorokhod 61', McKean '63, ..., Falkowski & Slominski '16, ...

Classical probabilistic examples for A and X:

- A is "time" t, a stoch. process of bounded variation, a fractional Brownian motion with Hurst index $H > 1/2$, ...
- X is a Brownian motion, martingale, Lévy process, semi-martingale, ...

Examples for L: $L = 0$ or adapted stoch. processes, "domains", ...

Applications:

- probability theory: construction of constrained stoch. processes, ...
- mathematical modeling: queueing theory, mathematical finance, ...

Pathwise reflected differential equations

Aim: pathwise reflected càdlàg differential equations, i.e.

$$Y_t = y + \int_0^t f_1(Y_s)\,\mathrm{d}A_s + \int_0^t f_2(Y_s)\,\mathrm{d}X_s + K_t, \quad t \in [0, T],$$

such that $Y_t \geq L_t$ and (b) hold.

Motivation:

- new well-posedness results (non-semimartingale structure, ...)
- pathwise stability results
- deeper understanding of equations
- ...

Keep in mind: e.g. $X = W$ is a Brownian motion.

$\Rightarrow X \in C^\alpha$ a.s. for $\alpha < 1/2$ and one expects $f_2(Y) \in C^\alpha$.

$\Rightarrow \int_0^t f_2(Y_s)\,\mathrm{d}X_s$ is in general not well-defined.

Outline

Setting $f_2 = 1$, let us first deal with

$$Y_t = y + \int_0^t f(Y_s)\,\mathrm{d}A_s + X_t + K_t, \quad t \in [0, T],$$

such that $Y_t \geq L_t$ and (b) hold, given

- $L \in D^p([0,T];\mathbb{R}^d)$ ($\sim$ càdlàg of finite p-variation),
- $A \in D^q([0,T];\mathbb{R}^d)$ for $q \in [1,2)$, $X \in D^p([0,T];\mathbb{R}^d)$ for $p \in (2,3)$.

Recall: $D^p([0,T];\mathbb{R}^d)$ denotes the space of all càdlàg paths $x\colon [0,T] \to \mathbb{R}^d$ of finite p-variation, i.e.

$$\|x\|_p := \left(\sup_{\mathcal{P}\subset[0,T]} \sum_{[u,v]\in\mathcal{P}} |X_v - X_u|^p \right)^{\frac{1}{p}} < \infty.$$

Need to define $\int_0^t f(Y_s)\,\mathrm{d}A_s$ for $Y \in D^p([0,T];\mathbb{R}^n)$.

For $x \in D^q([0,T];\mathbb{R}^d)$ and $y \in D^p([0,T];\mathcal{L}(\mathbb{R}^d;\mathbb{R}^n))$, the Young integral

$$\int_s^t y_r \,\mathrm{d}x_r := \lim_{|\mathcal{P}([s,t])|\to 0} \sum_{[u,v]\in\mathcal{P}([s,t])} y_u x_{u,v}, \quad s,t \in [0,T],$$

exists whenever $1/p + 1/q > 1$, and it comes with the estimate

$$\left| \int_s^t y_r \,\mathrm{d}x_r - y_s x_{s,t} \right| \leq C_{p,q} \|y\|_{p,[s,t)} \|x\|_{q,[s,t]},$$

for some constant $C_{p,q}$, see Young '36 and Friz & Zhang '18.

Note it is crucial here to take left-point Riemann sums.

Uniqueness and existence result

Consider the reflected Young differential equation

$$Y_t = y + \int_0^t f(Y_s)\,\mathrm{d}A_s + X_t + K_t, \quad t \in [0, T], \tag{1}$$

such that $Y_t \geq L_t$ and (b) hold.

Theorem (Allan, Liu, P. '20)

Let $f \in C_b^2$, $q \in [1,2)$ and $p \in [q,\infty)$ such that $1/p + 1/q > 1$. Let $y \in \mathbb{R}^n$, $A \in D^q([0,T];\mathbb{R}^d)$, $X \in D^p([0,T];\mathbb{R}^d)$ and $L \in D^p([0,T];\mathbb{R}^n)$ such that $y \geq L_0$.

Then, there exists a unique solution (Y, K) to the reflected Young differential equation (1).

Related works: Ferrante & Rovira '13, Falkowski & Slominski '15, ...

Skorokhod problem

Reflected differential equations are related to the Skorokhod problem.

Given $Y, L \in D([0,T];\mathbb{R}^n)$ be such that $Y_0 \geq L_0$. A solution to the Skorokhod problem is a pair $(Z,K) = \mathcal{S}(Y,L)$ such that

(a) $Z_t = Y_t + K_t \geq L_t$ for $t \in [0,T]$,

(b) $K_0 = 0$ and $K = (K^1, \dots, K^n)$, where K^i is non-decreasing function such that
$$\int_0^t (Z_s^i - L_s^i)\,\mathrm{d}K_s^i = 0, \quad t \in [0,T],$$
for every $i = 1, \dots, n$, where the latter integral is understood in the Lebesgue–Stietjes sense.

Note: There exists a unique solution to the Skorokhod problem.

(See e.g. Burdzy, Kang & Ramanan '09)

The associated Skorokhod map $\mathcal{S}$ is denoted by

$$\mathcal{S}\colon (Y,L) \mapsto (\mathcal{S}_1((Y,L)), \mathcal{S}_2((Y,L))) := (Z,K).$$

Note:

- $\mathcal{S}$ is continuous w.r.t. uniform norms,

  see Dupuis & Ishii '91, Dupuis & Ishii '93, Dupuis & Ramanan '99, ...
- $\mathcal{S}$ is not Lipschitz continuous w.r.t. Hölder norms,

  see Ferrante & Rovira '13.

Theorem (Falkowski & Slominski '15)

$\mathcal{S}\colon (Y,L) \mapsto (Z,K)$ is Lipschitz continuous w.r.t. p-variation, that is

$$\|Z-\tilde{Z}\|_p + \|K-\tilde{K}\|_p \leq C\Big(\|Y-\tilde{Y}\|_p + |Y_0-\tilde{Y}_0| + \|L-\tilde{L}\|_p + |L_0-\tilde{L}_0|\Big).$$

Proof of existence and uniqueness

For $t \in (0, T]$ we define the solution map

$$\mathcal{M}_t \colon D^p([0,t];\mathbb{R}^n) \to D^p([0,t];\mathbb{R}^n)$$

by

$$\mathcal{M}_t(Y) := \mathcal{S}_1\Big(y + \int_0^{\cdot} f(Y_r)\,\mathrm{d}A_r + X,\, L\Big).$$

Step 1:

- $\Rightarrow$ $\mathcal{M}_t$ is a contraction map provided the norms of A, X, L are small.
- $\Rightarrow$ $\exists!$ fixed point of the map $\mathcal{M}_t$ (Banach's fixed point theorem)
- $\Rightarrow$ $\exists!$ local solution to the reflected Young differential equation.

Step 2: Apply a pasting argument to construct a global solution.
Note one needs to treat the "large" jumps of the A, X, L per hand.

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Proposition (Allan, Liu, P. '20)

Let $f \in C_b^2$, $q \in [1,2)$ and $p \in [q,\infty)$ such that $1/p + 1/q > 1$.

Let (Y,K) and $(\tilde{Y},\tilde{K})$ be the unique solutions to the reflected Young differential equation given $y,\tilde{y} \in \mathbb{R}^n$, $A,\tilde{A} \in D^q([0,T];\mathbb{R}^d)$, $X,\tilde{X} \in D^p([0,T];\mathbb{R}^d)$ and $L,\tilde{L} \in D^p([0,T];\mathbb{R}^n)$ such that $y_0 \geq L_0$ and $\tilde{y}_0 \geq \tilde{L}_0$, respectively.

If $\|A\|_q \leq M$ and $\|\tilde{A}\|_q \leq M$ for some constant M, we have the estimates

$$\|Y - \tilde{Y}\|_p + \|K - \tilde{K}\|_p \leq C_{M,f}\Big(|y - \tilde{y}| + \|A - \tilde{A}\|_q + \|X - \tilde{X}\|_p + |L_0 - \tilde{L}_0| + \|L - \tilde{L}\|_p\Big)$$

for some constant $C_{M,f}$ depending M, $\|f\|_{C_b^2}$, p and d.

Consequences of these results:

- No semi-martingale structure is needed.
- new well-posedness results for Gaussian processes, Dirichlet processes, Markov processes, ...
- pathwise stability results w.r.t. the p-variation distance and also w.r.t. the Skorokhod J1 p-variation distance
- ...

$\Rightarrow$ The pathwise theory of reflected Young differential equations works equally well as the pathwise theory of classical Young differential equations.

Outline

Reflected rough differential equations (RDEs)

Coming back to reflected RDE

$$Y_t = y + \int_0^t f_1(Y_s)\,\mathrm{d}A_s + \int_0^t f_2(Y_s)\,\mathrm{d}X_s + K_t, \quad t \in [0, T],$$

such that $Y_t \geq L_t$ and (b) hold.

Rough input: $X \in D^p([0,T];\mathbb{R}^d)$ with $p \in (2,3)$.

$\Rightarrow Y \in D^p([0,T];\mathbb{R}^n)$

Many recent approaches: Besalú, Máquez-Carreras & Rovira '14, Aida '15, Aida '16, Castaing, Marine & De Fitte '17, Deya, Gubinelli, Hofmanová & Tindel '19, Richard, Tanre & Torres '19, Gassiat '20, Ananova '20, ...

We rely on càdlàg rough path theory developed by Friz & Shekhar '17, Friz & Zhang '18, Chevyrev & Friz '19, ...

A pair $\mathbf{X} = (X, \mathbb{X})$ is called a càdlàg p-rough path over $\mathbb{R}^d$ if

(i) $X \in D^p([0,T];\mathbb{R}^d)$ and $\mathbb{X} \in D^{\frac{p}{2}}(\Delta_T;\mathbb{R}^d)$,

(ii) $\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t}$ for $0 \leq s \leq u \leq t \leq T$.

A pair (Y, Y') is called controlled path with respect to X if

$$Y \in D^p([0,T];\mathbb{R}^n) \quad \text{and} \quad Y' \in D^p([0,T];\mathbb{R}^{d\times n})$$

satisfy

$$R^Y_{s,t} := Y_{s,t} - Y'_s X_{s,t} \in D^{\frac{p}{2}}(\Delta_T;\mathbb{R}^d).$$

The (forward) rough integral

$$\int_s^t Y_r \,\mathrm{d}\mathbf{X}_r := \lim_{|\mathcal{P}([s,t])|\to 0} \sum_{[u,v]\in\mathcal{P}([s,t])} Y_u X_{u,v} + Y'_u \mathbb{X}_{u,v}, \quad s,t \in [0,T],$$

exists and comes with the estimate

$$\Big| \int_s^t Y_r \,\mathrm{d}\mathbf{X}_r - Y_s X_{s,t} - Y'_s \mathbb{X}_{s,t} \Big| \leq C\Big(\|R^Y\|_{\frac{p}{2},[s,t)} \|X\|_{p,(s,t]} + \|Y'\|_{p,[s,t)} \|\mathbb{X}\|_{\frac{p}{2},(s,t]} \Big).$$

Note:

- $\Delta_t \Big(\int_0^\cdot Y_r \,\mathrm{d}\mathbf{X}_r \Big) = Y_{t-} \Delta X_t + Y'_{t-} \Delta \mathbb{X}_t,$
- Integration is a local Lipschitz continuous operator.
- classical RDE: existence, uniqueness, stability results, ...

Existence result - reflected RDEs

Consider the reflected RDE

$$Y_t = y + \int_0^t f(Y_s)\,\mathrm{d}\mathbf{X}_s + K_t, \quad t \in [0, T], \tag{2}$$

such that $Y_t \geq L_t$ and (b) hold.

Proposition (Allan, Liu, P. '20)

For $p \in (2,3)$ let $\mathbf{X} = (X, \mathbb{X})$ be a càdlàg p-rough path, $L \in D^p([0,T];\mathbb{R}^n)$ and $f \in C_b^3$. Then, for every $y \in \mathbb{R}^n$ with $y \geq L_0$ there exists a solution (Y, Y', K) to the reflected RDE (2) on $[0,T]$.

Related works (all for continuous rough paths):

- domains: Aida '15
- $L = 0$: Deya, Gubinelli, Hofmanova & Tindel '19
- one-dimensional: Richard, Tanre & Torres '19
- path-dependent RDEs: Aida '16 and Ananova '20

Proof - Existence

Recall

$$Y_t = y + \int_0^t f(Y_s)\,\mathrm{d}\mathbf{X}_s + K_t$$

$\Rightarrow$ the controlled structure looks like

$$\underbrace{Y_{s,t}}_{p\text{-var}} = \underbrace{f(Y_s)}_{p\text{-var}} X_{s,t} + \underbrace{R_{s,t}^{\int_0^\cdot f(Y_r)\,\mathrm{d}\mathbf{X}_r} + K_{s,t}}_{\frac{p}{2}\text{-var}}$$

$\Rightarrow$ Banach's fixed point argument would require

$$\|K - \tilde{K}\|_{\frac{p}{2}} \lesssim \|K - \tilde{K}\|_p$$

but this inequality does <u>not</u> hold, see Deya, Gubinelli, Hofmanová & Tindel '19.

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Observation: Since K is increasing, an interpolation arguments reveals

$$\|K-\tilde{K}\|_{\frac{p}{2}} \leq \|K-\tilde{K}\|_1^{\frac{2}{p}} \, \|K-\tilde{K}\|_p^{1-\frac{2}{p}}.$$

$\Rightarrow$ the "extended" Skorokhod map $\hat{\mathcal{S}}$ is locally Hölder continuity w.r.t. $\|\cdot\|_p + \|\cdot\|_{p/2}$ on the space of controlled paths $\mathcal{V}^p_X([0,t];\mathbb{R}^n)$.

Introduce the solution map $\mathcal{M}_t$ by

$$\mathcal{M}_t \colon \mathcal{V}^p_X([0,t];\mathbb{R}^n) \to \mathcal{V}^p_X([0,t];\mathbb{R}^n),$$

$$\text{via } \mathcal{M}_t(Y,Y') := \big(\mathcal{S}_1(y+Z,L), f(Y)\big) \quad \text{with } Z_u := \int_0^u f(Y_r)\,\mathrm{d}\mathbf{X}_r.$$

Proof - Existence

Step 1: For $t \in (0, T]$ small enough:

$\Rightarrow$ $\mathcal{M}_t$ is a continuous map between compact sets: compactness can be achieved since jumps are controlled by X & L ($\rightsquigarrow$ equi-regulated sets)

$\Rightarrow$ $\exists$ fixed point of the map $\mathcal{M}_t$ (Schauder's fixed point theorem)

$\Rightarrow$ $\exists$ solution to the reflected RDE on small intervals

Step 2: Apply a pasting argument to construct a global solution.
Note one needs to treat the "large" jumps of the X, L per hand.

Outline

Uniqueness for one-dimensional reflected RDEs

Note: There exists a linear 2-dim. RDE reflected at $L = 0$ with infinitely many solutions, see Gassiat '20.

Consider the one-dimensional reflected RDE

$$Y_t = y + \int_0^t f(Y_s)\,\mathrm{d}\mathbf{X}_s + K_t, \quad t \in [0, T], \tag{3}$$

such that $Y_t \geq L_t$ and (b) hold.

Theorem (Allan, Liu, P. '20)

Let $f \in C_b^3$, $p \in [2, 3)$, $\mathbf{X} = (X, \mathbb{X})$ be a càdlàg p-rough path, $L \in D^p([0, T]; \mathbb{R})$ and $y \in \mathbb{R}$ with $y \geq L_0$.

There exists at most one solution (Y, Y', K), with $Y' = f(Y)$, to the one-dimensional reflected RDE (3).

Uniqueness - proof

Related works for continuous rough path:

- $L = 0$: Deya, Gubinelli, Hofmanova & Tindel '19
- L with joint rough path $(\mathbf{X}, L)$: Richard, Tanre & Torres '19

Both works rely on a rough Grönwall's inequality from Deya, Gubinelli, Hofmanova & Tindel '19.

We proof of uniqueness by contradiction:

- Assume there two solutions Y, $\tilde{Y}$ s.t. $Y_a \neq \tilde{Y}_a$ for some $a \in (0, T]$.
- W.l.o.g. 0 is the last time when they are equal.

Case 1. There exists $t > 0$ s.t. $[0, t] \ni s \mapsto K_s - \tilde{K}_s$ is monotone.
Then

$$\|K - \tilde{K}\|_{\frac{p}{2},[0,t]} = \|K - \tilde{K}\|_{p,[0,t]},$$

which is precisely what we need for a contraction argument.

Uniqueness - proof

Case 2. Otherwise: $[0, t] \ni s \mapsto K_s - \tilde{K}_s$ is oscillating for every $t > 0$.

$\Rightarrow Y, \tilde{Y}$ must hit the barrier infinitely many times immediately after 0.

$\Rightarrow Y, \tilde{Y}$ either meet or jump over each other infinitely many times.

In both cases we obtain a contradiction.

Note: the contraction argument extends to the multidimensional case.

In this case, if uniqueness is lost, then both solutions $Y, \tilde{Y}$ must hit the barrier infinitely many times immediately afterwards.

Indeed, this is exactly what happens in Gassiat's example.

Conclusion

- reflected Young differential equations: existence, uniqueness, stability
- existence for reflected rough differential equations
- uniqueness for reflected one-dimensional RDEs

Thank you very much for your attention!

Reference:

Allan, A. L., Liu, C., and Prömel, D. J. (2020).
Càdlàg Rough Differential Equations with Reflecting Barriers.
Preprint arXiv:2008.00794.