Signature cumulants and generalized Magnus expansions

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FG6

Motivation: Moments and Cumulants

Let X be a r.v. s.t. $\mathbb{E}e^{\lambda X} < \infty$ for some $\lambda > 0$.

Definition

Moment-generating function:

$$\mu(z) := \mathbb{E}[e^{zX}] = \sum_{n=0}^{\infty} \mathbb{E}[X^n]\frac{z^n}{n!}, \quad z \leq \lambda$$

Cumulant-generating function:

$$\kappa(z) := \log \mu(z) = \sum_{n=1}^{\infty} \kappa_n \frac{z^n}{n!}, \quad z \leq \lambda.$$

Under some conditions, cumulants characterize the distribution of X, e.g. $X \sim \mathcal{N}(0, \sigma^2)$ iff

$$\kappa_1 = 0, \ \ \kappa_2 = \sigma^2, \ \ \kappa_3 = \kappa_4 = \cdots = 0.$$

Also, $X \sim \text{Poisson}(\lambda)$ iff

$$\kappa_n = \lambda, \quad n \geq 1.$$

Theorem (Leonov–Shiryaev (1959))

$$\kappa_n = \mathbb{E}[X^n] - \sum_{m=1}^{n-1} \binom{n-1}{m} \kappa_m \mathbb{E}[X^{n-m}]$$

Theorem (Speed (1983), Ebrahimi-Fard–Patras–T.–Zambotti (2018))

The following relation between moments and cumulants holds:

$$\mathbb{E}[X^n] = \sum_{\pi \in \mathcal{P}(n)} \prod_{B \in \pi} \kappa_{|B|}.$$

Also a multivariate version indexed by the same lattice.

Remark

Cumulants are also related to Wick products:

$$e^{zX - \kappa(z)} = \sum_{n=0}^{\infty} {:}X^n{:} \frac{z^n}{n!}$$

Motivation: Sine–Gordon model

Let K be a positive-semidefinite kernel on $\mathbb{R}^d$ and consider X a Gaussian field with covariance K, i.e. $\mathbb{E}[X(x)X(y)] = K(x,y)$.

Formally tilt the measure by setting

$$\tilde{\mathbb{P}}(\mathrm{d}X) := \frac{1}{Z}\exp\left(2\alpha\int \cos(\beta X(x))\,\mathrm{d}x\right)\mathbb{P}(\mathrm{d}X).$$

Since X is a distribution, this does not make sense.

Consider a regularised kernel K_t and the corresponding field $X_t(x)$ with $\mathbb{E}[X_s(x)X_t(y)] = K_{t\wedge s}(x,y)$.

This gives rise to the martingale

$$M_t := \int \cos(\beta X_t(x))\mathrm{e}^{\frac{\beta^2}{2}K_t(x,x)}\,\mathrm{d}x$$

One can show that $\mathbb{E}[\mathrm{e}^{\alpha M_t}]$ equals

$$\sum_{n=0}^{\infty}\frac{\alpha^n}{2^n n!}\sum_{\lambda\in\{+1,-1\}^n}\int \exp\left(-\beta^2\sum_{i<j}\lambda_i\lambda_j K_t(x_i,x_j)\right)\prod_{i=1}^{n}\mathrm{d}x_i.$$

Let $\mathbb{K}_t^{(n)}$ be the “martingale cumulants” and define

$$Z_t[f] := \mathbb{E}[f(X)\mathrm{e}^{\alpha M_t}]\mathrm{e}^{-\sum_{k=1}^{n-1}\frac{\alpha^{2k}}{(2k)!}\mathbb{K}_t^{(2k)}}.$$

Theorem (Lacoin–Rhodes–Vargas (2019))

For $\beta^2 < 2d$, the sequence Z_t has a limit as $t\to\infty$. Moreover,

$$\tilde{\mathbb{P}}(\mathrm{d}X) := \lim_{t\to\infty}\frac{1}{Z_t[1]}\mathrm{e}^{\alpha M_t-\sum_{k=1}^{n-1}\frac{\alpha^{2k}}{(2k)!}\mathbb{K}_t^{(2k)}}\,\mathbb{P}(\mathrm{d}X)$$

defines a probability measure.

Motivation: Diamond expansions

Let $\mathscr{S}$ be the space of semimartingales on a filtered probability space $(\mathcal{F}_t)_{t\geq 0}$.

Definition (Diamond product)

Let $X, Y \in \mathscr{S}$. Define

$$(X \diamond Y)_t(T) := \mathbb{E}_t[\langle X^c, Y^c\rangle_{t,T}] \in \mathscr{S}.$$

Theorem (Gatheral–Radoičić (2018), Friz–Gatheral–Radoičić (2020), Lacoin–Rhodes–Vargas (2019))

Let $A_T \in \mathcal{F}_T$ sufficiently integrable. Set $\mu_t(T) := \mathbb{E}_t[e^{zA_T}]$ and $\mathbb{K}_t(T) := \log \mu_t(T)$. If $\mathbb{K}_t(T) = z\mathbb{E}_t[A_T] + \sum_{n\geq 2} z^n \mathbb{K}_t^{(n)}(T)$, we have the recursion

$$\mathbb{K}_t^{(1)}(T) := \mathbb{E}_t[A_T]$$

$$\mathbb{K}_t^{(n+1)}(T) = \frac{1}{2}\sum_{k=1}^{n} (\mathbb{K}^{(k)} \diamond \mathbb{K}^{(n-k)})_t(T).$$

Proof.

Let $M_t := \mathbb{E}_t[A_T]$, $\Lambda_t^T = \sum_{k\geq 2} z^n \mathbb{K}_t^{(n)}(T)$. Then $e^{zM_t + \Lambda_t^T}$ is a martingale, so $zM_t + \Lambda_t^T + \frac{1}{2}\langle zM_t + \Lambda_t^T\rangle_t$ is also a martingale by Itô's formula. In particular

$$\mathbb{E}_t\left\{\mathbb{K}_T(T) + \frac{1}{2}\langle \mathbb{K}(T)\rangle_{t,T}\right\} = 0.$$

□

Signatures

$T(\!(\mathbb{R}^d)\!)$ is the completed tensor algebra over $\mathbb{R}^d$. Elements are formal tensor series

$$\mathbf{X} = \sum_{n=0}^{\infty} \mathbf{X}^{(n)}, \quad \mathbf{X}^{(n)} \in (\mathbb{R}^d)^{\otimes n},$$

and are multiplied according to

$$\mathbf{X}\mathbf{Y} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mathbf{X}^{(k)} \otimes \mathbf{Y}^{(n-k)}.$$

Sometimes word notation is also convenient: for $\mathbf{X} \in T(\!(\mathbb{R}^d)\!)$,

$$\mathbf{X} = \sum_{w} X^w w, \quad \mathbf{X}^{(n)} = \sum_{|w|=n} X^w w.$$

The symmetric algebra is $S(\!(\mathbb{R}^d)\!)$. There is a canonical projection $T(\!(\mathbb{R}^d)\!) \to S(\!(\mathbb{R}^d)\!)$, $\mathbf{X} \mapsto \hat{\mathbf{X}}$, and

$$\hat{X}^{i_1 \cdots i_n} = \frac{1}{n!} \sum_{\sigma \in \mathbb{S}_n} X^{i_{\sigma(1)} \cdots i_{\sigma(n)}}.$$

Definition

$$\begin{aligned}
\mathfrak{t} &:= \{\mathbf{X} \in T(\!(\mathbb{R}^d)\!) : \mathbf{X}^{(0)} = 0\} \\
\mathcal{T} &:= \{\mathbf{X} \in T(\!(\mathbb{R}^d)\!) : \mathbf{X}^{(0)} = 1\} = \exp(\mathfrak{t}) \\
\mathfrak{g} &:= \mathrm{FLA}(\!(\mathbb{R}^d)\!) \\
\mathcal{G} &:= \exp(\mathfrak{g}).
\end{aligned}$$

Definition

Let X be of bounded variation, its signature is the series

$$\mathrm{Sig}(X)_{s,t} = 1 + \sum_{n=1}^{\infty} \int \cdots \int_{s<u_1<\cdots<u_n<t} \mathrm{d}X_{u_1} \otimes \cdots \otimes \mathrm{d}X_{u_n} \in \mathcal{G}$$

Theorem (Hausdorff (1906))

Let $\Omega_{s,t} := \log \mathrm{Sig}_{s,t}(X) \in \mathfrak{g}$. *Then*

$$\mathrm{d}\Omega_{s,t} = H(-\operatorname{ad}\Omega_{s,t})\,\mathrm{d}X_t, \quad \Omega_{s,s} = 0$$

where $H(z) := \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$.

Tensor-valued semimartingales

Definition (FHT (2021))

A tensor-valued semimartingale is a series $\mathbf{X}$ where X^w is a semimartingale for all w, and $X^\varnothing$ is constant. The space is denoted by $\mathscr{S}((\mathbb{R}^d))$.

Definition (see e.g. Protter's book)

Square bracket:

$$[X,Y] := XY - \int X_- \,\mathrm{d}Y - \int Y_- \,\mathrm{d}X$$

Angle bracket:

$$\langle X^c, Y^c \rangle := [X,Y] - \sum_{s \le \cdot} \Delta X_s \Delta Y_s$$

Definition (see e.g. Protter's)

For $q \in [1,\infty)$, the space $\mathscr{H}^q \subset \mathscr{S}$ are all $X \in \mathscr{S}$ such that

$$\|X\|_{\mathscr{H}^q} := \inf_{X=M+A} \left\| [M]_T^{1/2} + \int_0^T |\mathrm{d}A_u| \right\|_{L^q} < \infty.$$

Definition (FHT (2021))

Outer bracket:

$$[\![\mathbf{X}, \mathbf{Y}]\!] := \sum_{u,v} [X^u, Y^v]\, u \otimes v$$

Inner bracket:

$$\langle \mathbf{X}^c, \mathbf{Y}^c \rangle := \sum_w \left(\sum_{uv=w} \langle X^{u,c}, Y^{v,c} \rangle \right) w$$

Generalized signatures

Definition (Friz–Shekhar (2017), FHT (2021))

Let $\mathbf{X} \in \mathscr{S}((\mathfrak{t}))$. Its generalized signature $\mathrm{Sig}(\mathbf{X})_{s,t}$ is the unique solution to the Marcus equation

$$\mathbf{S}_{s,t} = 1 + \int_{(s,t]} \mathbf{S}_{s,u-}\, \mathrm{d}\mathbf{X}_u + \frac{1}{2}\int_s^t \mathbf{S}_{s,u-}\, \mathrm{d}\langle \mathbf{X}^c\rangle_u + \sum_{s<u\le t} \mathbf{S}_{s,u-}(\exp(\Delta\mathbf{X}_u) - 1 - \Delta\mathbf{X}_u) =: 1 + \int_s^t \mathbf{S}_{s,u-} \circ \mathrm{d}\mathbf{X}_u.$$

Proposition (Chen (1953), Lyons (1998), FHT (2021))

Let $\mathbf{X} \in \mathscr{S}((\mathfrak{t}))$ *and* $s, u, t \in [0, T]$.

$$\mathrm{Sig}(\mathbf{X})_{s,u}\, \mathrm{Sig}(\mathbf{X})_{u,t} = \mathrm{Sig}(\mathbf{X})_{s,t}$$

Proposition (FHT (2021))

For $\mathbf{X} \in \mathscr{S}((\mathfrak{t}))$,

$$\mathrm{Sig}(\mathbf{X})_{s,\cdot} \in \mathscr{S}((\mathcal{T}))$$

for all $s \in [0, T]$.

Definition (Lyons–Victoir (2004))

$$\boldsymbol{\mu}_t(T) := \mathbb{E}_t\, \mathrm{Sig}(\mathbf{X})_{t,T} \in \mathcal{T}$$
$$\boldsymbol{\kappa}_t(T) := \log \boldsymbol{\mu}_t(T) \in \mathfrak{t}$$

Theorem (Bonnier–Oberhauser (2019))

$$\boldsymbol{\mu}_t(T)^w = \sum_{\pi \in \mathrm{OP}(|w|)} \frac{1}{|\pi|!} \boldsymbol{\kappa}_t(T)^\pi.$$

Generalized signatures

From now on, $\mathfrak{t}_N, \mathcal{T}_N$, etc. denote truncated versions of the corresponding spaces. Moreover π_N will denote the corresponding projection.

Definition

For $N \geq 1$, the space $\tilde{\mathscr{H}}^{q,N}$ consists of all $\mathbf{X} \in \mathscr{S}(\mathfrak{t}_N)$ such that

$$\|\mathbf{X}\|_{\tilde{\mathscr{H}}^{q,N}} := \sum_{n=1}^{N} \|\mathbf{X}^{(n)}\|^{1/n}_{\mathscr{H}^{qN/n}} < \infty.$$

In particular, if $\mathbf{X} = (0, X, 0, \dots) \in \mathbb{R}^d \subset \mathfrak{t}_N$, then $\|\mathbf{X}\|_{\tilde{\mathscr{H}}^{q,N}} = \|X\|_{\mathscr{H}^{qN}}$.

Definition (FHT (2021))

For $\mathbf{X}^{(n)} \in \mathscr{S}((\mathbb{R}^d)^{\otimes n})$ and $q \in [1, \infty)$,

$$\|\mathbf{X}^{(n)}\|_{\mathscr{H}^q} := \inf_{\mathbf{X}^{(n)}=\mathbf{M}+\mathbf{A}} \left\| |[\mathbf{M}]_T|^{1/2} + |\mathbf{A}|_{1\text{-var};[0,T]} \right\|_{L^q}.$$

Definition

$$\tilde{\mathscr{H}}^{\infty-} := \{\mathbf{X} \in \mathscr{S}(\!(\mathbb{R}^d)\!) : \pi_N \mathbf{X} \in \tilde{\mathscr{H}}^{q,N}, \forall q \in [1, \infty), \forall N \in \mathbb{N}\}.$$

Generalized signatures

Theorem

Let $q \in [1, \infty)$ and $N \in \mathbb{N}$. There are constants $c, C > 0$ depending on q and N, such that for all $\mathbf{X} \in \mathscr{S}(\mathrm{t}_N)$ we have

$$c\|\mathbf{X}\|_{\mathscr{H}^{q,N}} \leq \|\mathrm{Sig}(\mathbf{X})\|_{\mathscr{H}^{q,N}} \leq C\|\mathbf{X}\|_{\mathscr{H}^{q,N}}.$$

Corollary (FHT (2021))

If $\mathbf{X} \in \tilde{\mathscr{H}}^{\infty-}$, then $\mathrm{Sig}(\mathbf{X}) \in \tilde{\mathscr{H}}^{\infty-}$ and

$$\boldsymbol{\mu}.(T) \in \mathscr{S}((\mathcal{T})), \quad \boldsymbol{\kappa}.(T) \in \mathscr{S}((\mathrm{t})).$$

Corollary (Friz–Victoir (2006), Chevyrev–Friz (2019))

Let M be a martingale, then the enhanced BDG inequality

$$\max_{i=1,\dots,N} \left\| \sup_{0\leq t\leq T} \left|\mathrm{Sig}(M)^{(n)}_{0,t}\right| \right\|^{1/n}_{L^{qN/n}} \leq C \left\| |[M]_T|^{1/2} \right\|_{L^{qN}}$$

holds.

Main result

Theorem (FHT (2021))

For a sufficiently integrable $\mathfrak{t}$-valued semimartingale $\mathbf{X}$, $\boldsymbol{\kappa}_t(T)$ is the unique solution to the functional equation

$$\boldsymbol{\kappa}_t = \mathbb{E}_t\Bigg\{\int_{(t,T]} H(\operatorname{ad}\boldsymbol{\kappa}_{u-})(\mathrm{d}\mathbf{X}_u) + \frac{1}{2}\int_t^T H(\operatorname{ad}\boldsymbol{\kappa}_{u-})(\mathrm{d}\langle\mathbf{X}^c\rangle_u)$$
$$+\frac{1}{2}\int_t^T H(\operatorname{ad}\boldsymbol{\kappa}_{u-})Q(\operatorname{ad}\boldsymbol{\kappa}_{u-},\operatorname{ad}\boldsymbol{\kappa}_{u-})(\mathrm{d}[\![\boldsymbol{\kappa}]\!]^c_u) + \int_t^T H(\operatorname{ad}\boldsymbol{\kappa}_{u-})\circ(\operatorname{id}\odot G(\operatorname{ad}\boldsymbol{\kappa}_{u-}))(\mathrm{d}[\![\mathbf{X},\boldsymbol{\kappa}]\!]^c_u)$$
$$+\sum_{t<u\le T}\big(H(\operatorname{ad}\boldsymbol{\kappa}_{u-})(\exp(\Delta\mathbf{X}_u)\exp(\boldsymbol{\kappa}_u)\exp(-\boldsymbol{\kappa}_{u-})-1-\Delta\mathbf{X}_u)-\Delta\boldsymbol{\kappa}_u\big)\Bigg\}$$

$$G(z) := \frac{\mathrm{e}^z-1}{z} = \sum_{k=0}^{\infty}\frac{z^n}{(n+1)!},\qquad Q(z,\tilde z) := \sum_{n,m=0}^{\infty}\frac{z^n\odot\tilde z^m}{(n+1)!m!(n+m+2)},\qquad U\odot V(\mathbf{X}\otimes\mathbf{Y}) = U(\mathbf{X})V(\mathbf{Y}).$$

Main result: Recursion

Corollary (FHT (2021))

The graded components $\boldsymbol{\kappa}_t^{(n)}(T)$ satisfy the recursion

$$\boldsymbol{\kappa}_t^{(1)}(T) = \mathbb{E}_t[\mathbf{X}_{t,T}^{(1)}]$$

$$\boldsymbol{\kappa}_t^{(n)}(T) = \mathbb{E}_t[\mathbf{X}_{t,T}^{(n)}] + \sum_{k=1}^{n} (\mathbf{X}^{(k)} \diamond \mathbf{X}^{(n-k)})_t(T) + \sum_{I \vdash n} \mathbb{E}_t[\Omega(I) + \mathbb{Q}(I) + \mathbb{C}(I) + \mathbb{J}(I)]$$

where

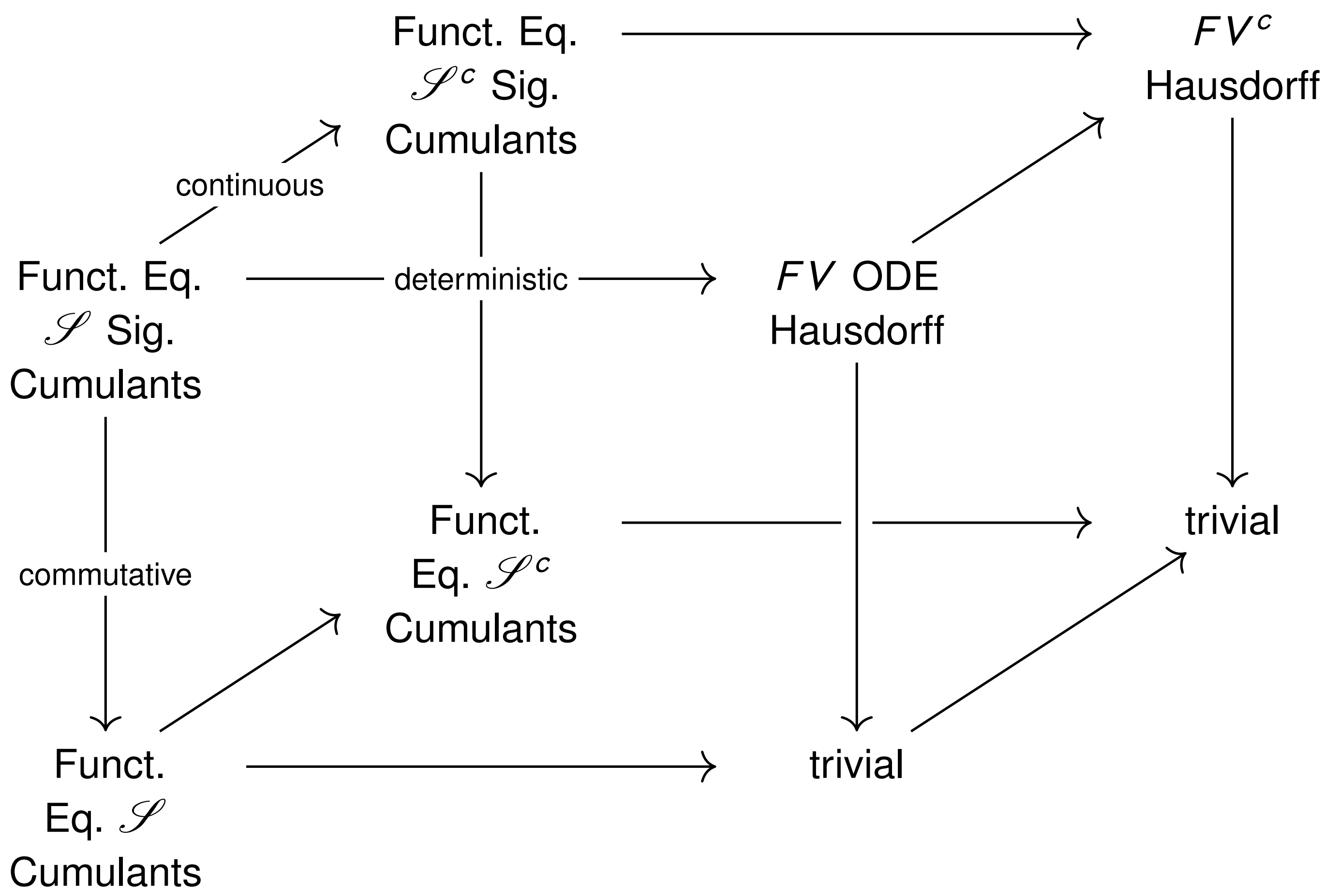
$$\Omega(I) = \frac{1}{k!} \int_{(t,T]} \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_2)} \cdots \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_k)} (\mathrm{d}\mathbf{X}^{(i_1)})$$

$$\mathbb{Q}(I) = \frac{1}{k!} \sum_{m=2}^{k} \binom{n-1}{m-1} \int_t^T \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_3)} \cdots \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_m)} \odot \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_{m+1})} \cdots \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_k)} (\mathrm{d}[\![\boldsymbol{\kappa}^{(i_1)}, \boldsymbol{\kappa}^{(i_2)}]\!]_u^c)$$

$$\mathbb{C}(I) = \frac{1}{(k-1)!} \int_t^T (\mathrm{id} \odot \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_3)} \cdots \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_k)})(\mathrm{d}[\![\mathbf{X}^{(i_1)}, \boldsymbol{\kappa}^{(i_2)}]\!]_u^c)$$

$$\mathbb{J}(I) = \sum_{t<u\leq T} \left(\sum_{1\leq m\leq j\leq k} (-1)^{k-j} \frac{\Delta \mathbf{X}_u^{(i_1)} \cdots \Delta \mathbf{X}_u^{(i_m)} \boldsymbol{\kappa}_u^{(i_{m+1})} \cdots \boldsymbol{\kappa}_u^{(i_j)} \boldsymbol{\kappa}_{u-}^{(i_{j+1})} \cdots \boldsymbol{\kappa}_{u-}^{(i_k)}}{m!(m-j)!(k-j)!} - \frac{1}{k!} \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_2)} \cdots \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_k)} (\Delta \boldsymbol{\kappa}_u^{(i_1)}) \right)$$

Main result: Overwiew



Consequences: Brownian motion

Let B be a standard BM and let $\mathrm{d}X_t = \sigma(t)\,\mathrm{d}B_t$.

Corollary (Fawcett (2002), FHT (2021))

The signature cumulants of X satisfy the functional equation

$$\boldsymbol{\kappa}_t(T) = \int_t^T H(\operatorname{ad}\boldsymbol{\kappa}_u)(a(u))\,\mathrm{d}u, \quad a = \sigma\sigma^\top.$$

In particular, if $X = B$, i.e. $\sigma = I = \sum_{i=1}^d ii$ we recover Fawcett's formula

$$\boldsymbol{\kappa}_t(T) = \frac{1}{2}\sum_{i=1}^d (T-t)ii, \qquad \mathbb{E}_t \operatorname{Sig}(B)_{t,T} = \exp\left(\frac{1}{2}\sum_{i=1}^d (T-t)ii\right)$$

Theorem (Lyons–Ni (2015), FHT (2021))

Let $\Gamma \subset \mathbb{R}^d$ bounded, regular domain and τ_Γ the first exit time of a BM B.
The signature cumulants $\boldsymbol{\kappa}_t = \log \mathbb{E}_t[\operatorname{Sig}(B)_{t\wedge\tau_\Gamma,\tau_\Gamma}]$ up to the first exit time from Γ have the form $\kappa_t = 1_{\{t<\tau_\Gamma\}}\mathbf{F}(B_t)$ where

$$-\Delta\mathbf{F}(x) = \sum_{i=1}^d H(\operatorname{ad}\mathbf{F}(x))\Big(ii + Q(\operatorname{ad}\mathbf{F}(x))(\partial_i\mathbf{F}(x)^2) + 2iG(\operatorname{ad}\mathbf{F}(x))(\partial_i\mathbf{F}(x))\Big)$$

with boundary condition $\mathbf{F}|_{\partial\Gamma} = 0$.

Consequences: Time-inhomogeneous Lévy processes

Suppose $X \in \mathscr{S}(\mathbb{R}^d)$ is an Itô semimartingale with independent increments. Then

$$X_t = \int_0^t b(u)\,\mathrm{d}u + \int_0^t \sigma(u)\,\mathrm{d}B_u + \int_{(0,t]}\int_{|x|\le 1} x(\mu^X - \nu)(\mathrm{d}u,\mathrm{d}x) + \int_{(0,t]}\int_{|x|>1} x\mu^X(\mathrm{d}u,\mathrm{d}x).$$

where $b \in L^1$, $\sigma \in L^2$, μ^X is an independent inhomogeneous Poisson random measure with intensity measure ν, such that $\nu(\mathrm{d}u,\mathrm{d}x) = K_u(\mathrm{d}x)\,\mathrm{d}u$ and K_u are Lévy measures with

$$\int_{\mathbb{R}^d}(|x|^2 \wedge 1)K_u(\mathrm{d}x) < \infty, \quad \int_0^T\int_{|x|>1}|x|^n K_u(\mathrm{d}x)\mathrm{d}u < \infty.$$

Corollary (FHT (2021), Friz–Shekhar (2017))

We have that $X \in \mathscr{H}^{\infty-}$ and the signature cumulants satisfy

$$\boldsymbol{\kappa}_t = \int_t^T H(\mathrm{ad}\,\boldsymbol{\kappa}_u)(\mathfrak{h}(u))\,\mathrm{d}u,$$

where

$$\mathfrak{h}(u) := b(u) + \frac{1}{2}a(u) + \int_{\mathbb{R}^d}(\exp(x) - 1 - x1_{|x|\le 1})K_u(\mathrm{d}x), \qquad a = \sigma.\sigma^\top.$$

Consequences: Symmetrization

Theorem (FHT (2021))

We have,

$$\widehat{\mathrm{Sig}(\mathbf{X})}_{s,t} = \exp\big(\hat{\mathbf{X}}_T - \hat{\mathbf{X}}_t\big)$$

Moreover, if $\mathbf{X} = (0, X, 0, \dots)$,

$$\hat{\boldsymbol{\mu}}_t(T) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}_t[(X_T - X_t)^{\otimes n}].$$

Theorem (Fukasawa–Matsushita (2020), FHT (2021))

Let $\tilde{\mathbf{X}} \in \mathscr{S}(\hat{\mathrm{t}})$ *and* $\mathbb{K}_t(T) := \log \mathbb{E}_t \exp(\tilde{\mathbf{X}}_T) = \tilde{\mathbf{X}}_t + \tilde{\boldsymbol{\kappa}}_t(T)$*. Then*

$$\mathbb{K}_t(T) = \frac{1}{2}(\mathbb{K} \diamond \mathbb{K})_t(T) + \sum_{t<u\leq T} \mathbb{E}_t[\exp(\Delta\mathbb{K}_u) - 1 - \Delta\mathbb{K}_u].$$

In particular, the multiviariate martingale cumulants of a continuous semimartingale X *satisfy the recursion*

$$\mathbb{K}_t^{(1)}(T) = \mathbb{E}_t(X_T)$$

$$\mathbb{K}_t^{(n+1)}(T) = \frac{1}{2} \sum_{k=1}^{n} (\mathbb{K}^{(k)} \diamond \mathbb{K}^{(n-k)})_t(T)$$

Consequences: Hausdorff and Magnus

Theorem (Hausdorff (1906), FHT (2021))

Assume $\mathbf{X}$ *is a deterministic càdlàg path of bounded variation. The log-signature* $\Omega_t(T) := \log \mathrm{Sig}(\mathbf{X})_{t,T}$ *satisfies*

$$\Omega_t = \int_{(t,T]} H(\operatorname{ad}\Omega_{u-})(\mathrm{d}\mathbf{X}^c_u) + \sum_{t<u\le T} \int_0^1 \Psi(\exp(\theta \operatorname{ad}\Delta\mathbf{X}_u) \circ \exp(\operatorname{ad}\Omega_u))(\Delta\mathbf{X}_u)\,\mathrm{d}\theta,$$

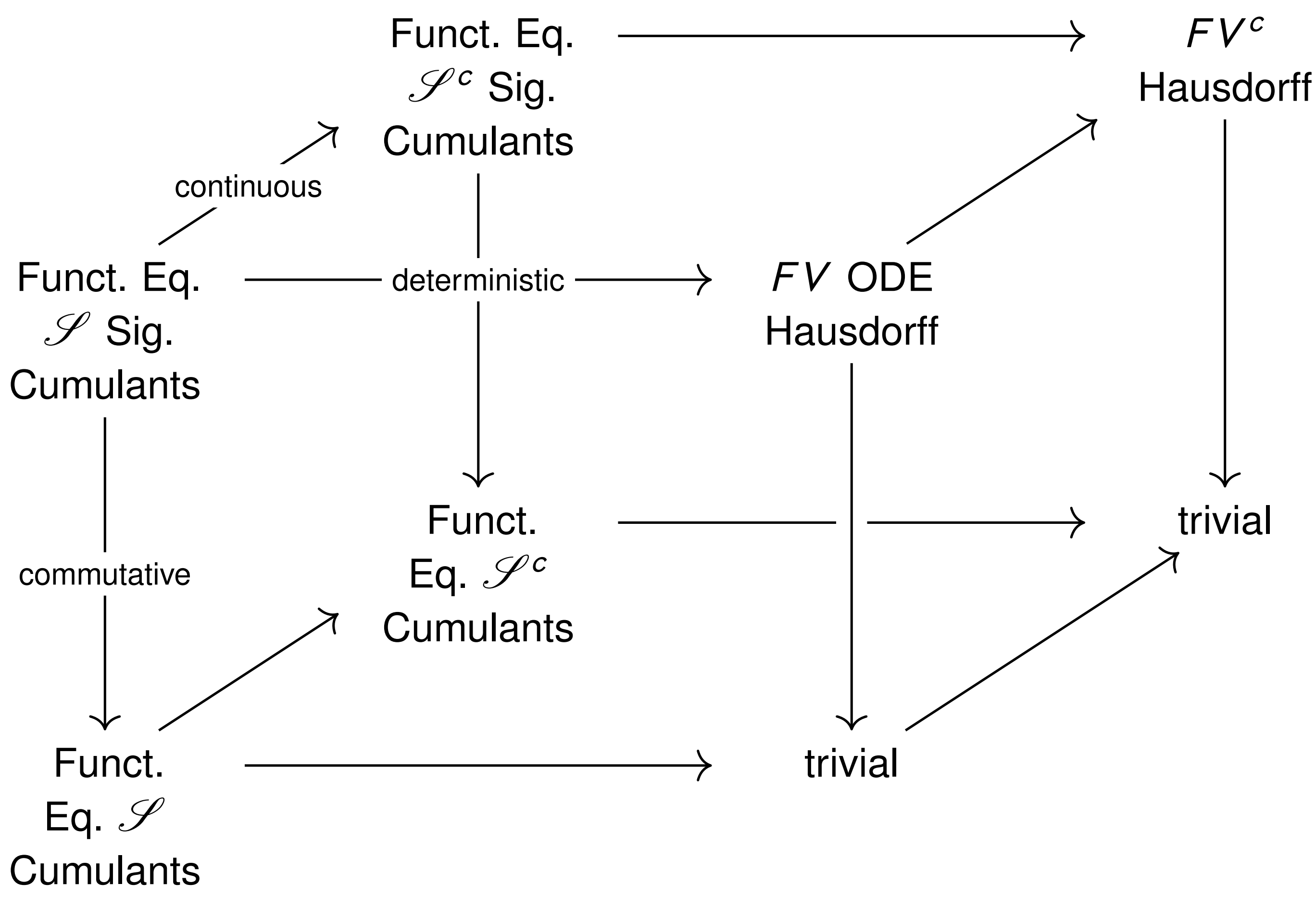
where

$$\Psi(z) := H(\ln z) = \frac{\ln z}{z-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(z-1)^n.$$

Corollary (FHT (2021))

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathfrak{t}$*. If* $\mathrm{BCH}(\mathbf{x}_1, \ldots, \mathbf{x}_n) := \log(\exp(\mathbf{x}_1)\cdots\exp(\mathbf{x}_n))$ *then*

$$\mathrm{BCH}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum_{k=1}^{n} \int_0^1 \Psi(\exp(\theta \operatorname{ad}\mathbf{x}_k) \circ \cdots \circ \exp(\operatorname{ad}\mathbf{x}_n))(\mathbf{x}_k)\,\mathrm{d}\theta$$


Funct. Eq. $\mathscr{S}^c$ Sig. Cumulants
FV^c Hausdorff
continuous
Funct. Eq. $\mathscr{S}$ Sig. Cumulants
deterministic
FV ODE Hausdorff
Funct. Eq. $\mathscr{S}^c$ Cumulants
trivial
commutative
Funct. Eq. $\mathscr{S}$ Cumulants
trivial

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