Convergence rates for the occupation measure of fractional Brownian motion

Dario Trevisan

Università degli Studi di Pisa

Pathwise Stochastic Analysis and Applications

CIRM, March 8-12 2021

Joint with F. Mattesini and M. Huesmann (U. Muenster)

Motivation

The empirical occupation measure of a Brownian path

$$\int f d\mu_T = \int_0^T f(B_t)dt$$

is quite common, for example

- in regularization by noise (Gubinelli, Catellier...): a key tool is that

  $$x \mapsto \int_0^T f(x + B_t)dt$$

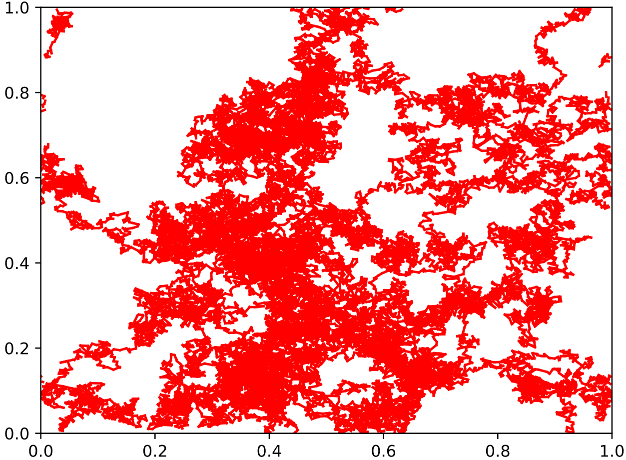
  is more regular than f itself (small T's here are relevant).
- in the study of covering times (Aldous, Dembo-Peres-Rosen-Zeitouni): how large should T be so that

  $$\mathrm{supp}(\mu_T) \text{ is } \varepsilon\text{-close to every point}$$

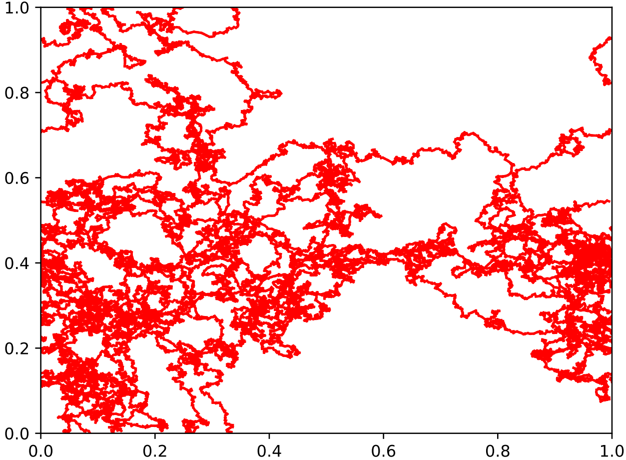
  (considering a Brownian motion on the torus $\mathbb{T}^d$)

Question: what happens if we consider a fractional Brownian motion?

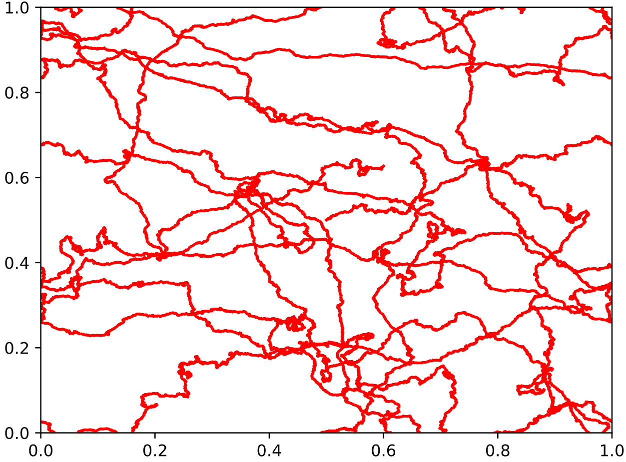
Simulations: H=0.5, T=1



Simulations: H=0.7, T=5



Simulations: H=0.9, T=10



As a variant of the covering time problem we can measure an average distance using the Kantorovich-Wasserstein optimal transport distance:

$$W_1(\mu_T, T\mathcal{L}^d_{\mathbb{T}^d}) = \inf_{\pi} \int_{\mathbb{T}^d \times \mathbb{T}^d} \mathrm{dist}_{\mathbb{T}^d}(x, y) d\pi(x, y),$$

where π are joint measures such that the marginals are respectively μ_T and $T\mathcal{L}^d_{\mathbb{T}^d}$.

The dual formulation reads

$$W_1(\mu_T, T\mathcal{L}^d_{\mathbb{T}^d}) = \sup_{\mathrm{Lip}(f) \leq 1} \left\{ \int_0^T f(B_s) ds - T \int_{\mathbb{T}^d} f(y) dy \right\},$$

i.e., we measure how close Lipschitz functions are to constants after averaging w.r.t. the occupation measure.

Problem

Estimate $W_1(\mu_T, T\mathcal{L}^d_{\mathbb{T}^d})$ as $T \to \infty$.

The random Euclidean bipartite matching problem

Another motivation is to study a variant of the random Euclidean bipartite matching problem, that reads: let n be i.i.d. uniformly distributed points $(X_i)_{i=1}^n$ on $\mathbb{T}^d$ and

$$\text{replace } \mu_T = \int_0^T \delta_{B_t} dt \quad \text{with the empirical measure} \quad \nu_n = \sum_{i=1}^n \delta_{X_i}.$$

Intuition: the typical distance is $n^{-\frac{1}{d}}$ (as in a regular grid).
But there are fluctuations (CLT)!

The asymptotic behaviour of $W_1(\nu_n, n\mathcal{L}^d_{\mathbb{T}^d})$ depends on the dimension d:

$$W_1(\nu_n, n\mathcal{L}^d_{\mathbb{T}^d}) \sim \begin{cases} n \cdot n^{-\frac{1}{2}} & \text{for } d = 1 \\ n \cdot \sqrt{\frac{\log n}{n}} & \text{for } d = 2 \\ n \cdot n^{-\frac{1}{d}} & \text{for } d \geq 3. \end{cases}$$

An open problem is to prove that the limit (after rescaling) exists for $d = 2$.

Related literature: the diffusion case

F.Y. Wang and J-X. Zhu (2019) consider diffusion processes on Riemannian Manifolds (thus including the case $H = 1/2$ on the torus).

Their main result is (actually for the W_2 distance)

$$W_1(\mu_T, T\mathcal{L}^d_{\mathbb{T}^d}) \sim \begin{cases} T \cdot T^{-\frac{1}{2}} & \text{for } d \leq 3 \\ T \cdot \sqrt{\frac{\log T}{T}} & \text{for } d = 4 \\ T \cdot T^{-\frac{1}{d-2}} & \text{for } d \geq 5. \end{cases}$$

Intuition:

- $\operatorname{supp} \mu_T$ is (roughly) 2-dimensional
- put n^{d-2} horizontal planes in $\mathbb{T}^d$ gives a distance of order $\frac{1}{n}$
- each plane has measure (area) 1, thus $n^{d-2} = T$
- typical distance is $\sim T^{-\frac{1}{d-2}}$,

But again there are fluctuations!

Our result

For $H \in (0,1)$ a fBM with Hurst index H on $\mathbb{T}^d$ is constructed as

d independent real fBM $(B_t^1, B_t^2, \dots, B_t^d)_{t \geq 0}$ projected on $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$.

Theorem (Huesmann, Mattesini, T.)

Let $H \in (0,1)$, consider a fBM with Hurst index H on $\mathbb{T}^d$ with empirical measure μ_T. Then,

$$\mathbb{E}\left[W_1(\mu_T, T\mathcal{L}^d_{\mathbb{T}^d}) \right] \sim \begin{cases} T \cdot T^{-\frac{1}{2}} & \text{for } d < \frac{1}{H} + 2 \\ T \cdot \sqrt{\frac{\log T}{T}} & \text{for } d = \frac{1}{H} + 2 \\ T \cdot T^{-\frac{1}{d - 1/H}} & \text{for } d > \frac{1}{H} + 2. \end{cases}$$

Notice that (formally) $H = \infty$ gives back the bipartite matching problem!

Some open problems

Comparison with the random bipartite matching problems leads to the following questions (that we have not addressed).

1. Limits. For $d < \frac{1}{H} + 2$ it should be not difficult to prove that

   $$\lim_{T \to \infty} \mathbb{E}\left[W_1(\mu_T, T\mathcal{L}^d_{\mathbb{T}^d}) \right] / \sqrt{T} \quad \text{exists.}$$

   If $d > \frac{1}{H} + 2$ it may be feasible.
   For $d = \frac{1}{H} + 2$ it should be challenging (open also for the matching problem).
2. Rates for W_p instead of W_1. In fact our proof covers $p \in [1, 4]$ (with same rate). Higher p's require more computational effort or better ideas?
   The case $p = \infty$ is related to covering times.
   The case $p \in (0, 1)$ may also be interesting.
3. Case $H = 1$. Pick a random direction (with random velocity) and follow the geodesics. The case $d = 3$ is open (not clear if the logarithm must be there)

Sketch of proof

We follow an established route for the bipartite matching problem – first proposed by S. Caracciolo and collaborators.

We combine tools from analysis and probability:

1. Optimal Transport Theory and PDE's
2. Fourier analysis
3. Gaussian processes

The main novelty are (fourth order) moment bounds on the Fourier transform of the occupation measure

$$\hat{\mu}_T(m) = \int_0^T \exp\left(2\pi i m \cdot B_s\right) ds.$$

Optimal transport bounds

The Wasserstein distance between two measures μ, ν on the line $\mathbb{R}$ can be computed via cumulative distribution functions (Dall'Aglio):

$$W_1(\mu, \nu) = \int_{-\infty}^{+\infty} |F_\mu(t) - F_\nu(t)| dt.$$

Are there similar expressions (or at least bounds) for $d > 1$?

Lemma

Given two measures (with smooth densities) μ, ν on $\mathbb{T}^d$, one has the upper bound

$$W_1(\mu, \nu) \leq \int_{\mathbb{T}^d} |\nabla \Delta^{-1}(\mu - \nu)|$$

and the lower bound

$$W_1(\mu, \nu) \geq \sup_{M>0} \left\{ \frac{1}{M} \int_{\mathbb{T}^d} |\nabla \Delta^{-1}(\mu - \nu)|^2 - \frac{C}{M^3} \int_{\mathbb{T}^d} |\nabla \Delta^{-1}(\mu - \nu)|^4 \right\},$$

where $C > 0$ is a constant depending on d only.

Problem: we cannot apply the bounds to a singular measure, e.g. $\mu = \mu_T$, they may produce diverging quantities.

We introduce a smoothing operator $\rightarrow$ the heat semigroup $P_\varepsilon \mu$. Key inequalities

$$W_1(\mu, P_\varepsilon \mu) \leq C\sqrt{\varepsilon}\mu(\mathbb{T}^d), \quad \text{with } C = C(d),$$

and

$$W_1(\mu, \nu) \geq W_1(P_\varepsilon \mu, P_\varepsilon \nu).$$

Lemma

Upper bound:

$$W_1(\mu_T, T\mathcal{L}^d_{\mathbb{T}^d}) \leq \inf_{\varepsilon > 0} C\sqrt{\varepsilon}T + \int_{\mathbb{T}^d} |\nabla \Delta^{-1} P_\varepsilon(\mu_T - T)|$$

and lower bound:

$$\begin{aligned} &W_1(\mu_T, T\mathcal{L}^d_{\mathbb{T}^d}) \\ &\geq \sup_{M, \varepsilon > 0} \left\{ \frac{1}{M} \int_{\mathbb{T}^d} |\nabla \Delta^{-1} P_\varepsilon(\mu_T - T)|^2 - \frac{C}{M^3} \int_{\mathbb{T}^d} |\nabla \Delta^{-1} P_\varepsilon(\mu_T - T)|^4 \right\}, \end{aligned}$$

where $C > 0$ is a constant depending on d only.

Some Fourier Analysis

To conveniently bound the terms $\int_{\mathbb{T}^d} |\nabla \Delta^{-1} P_\varepsilon(\mu - \nu)|^q$ we use Fourier series. We use the isometry

$$\int_{\mathbb{T}^d} |f|^2 = \sum_{m \in \mathbb{Z}^d} |\hat{f}(m)|^2$$

as well as

$$\int |f|^4 = \sum_{m_1+m_2+m_3+m_4=0} \hat{f}(m_1)\hat{f}(m_2)\hat{f}(m_3)\hat{f}(m_4).$$

We obtain

$$\int_{\mathbb{T}^d} |\nabla \Delta^{-1} P_\varepsilon(\mu_T - 1)|^2 = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} e^{-\varepsilon |m|^2} \frac{|\hat{\mu}(m)|^2}{|m|^2}.$$

and

$$\int_{\mathbb{T}^d} |\nabla \Delta^{-1} P_\varepsilon(\mu - 1)|^4 = \sum_{\substack{m_1+m_2+m_3+m_4=0 \\ m_i \neq 0}} \prod_{i=1}^{4} \frac{m_i}{|m_i|^2} e^{-\varepsilon\ |m_i|^2} \hat{\mu}(m_i)$$

Moment bounds

So far everything was valid for a general path in $\mathbb{T}^d$. Probability enters when we take expectations. We have

$$|\hat{\mu}_T(m)|^2 = \int_0^T \int_0^T \exp\left(2\pi i m(B_{s_2} - B_{s_1})\right) ds_1 ds_2$$

and

$$\hat{\mu}_T(m_1)\hat{\mu}_T(m_2)\hat{\mu}_T(m_3)\hat{\mu}_T(m_4) =$$
$$= \int_0^T \int_0^T \int_0^T \int_0^T \exp\left(2\pi i \sum_{j=1}^{4} m_j B_{s_j}\right) ds_1 ds_2 ds_3 ds_4$$

Exchanging expectations and integrals we find explicit formulas, e.g., for the second moment

$$\mathbb{E}\left[\exp\left(2\pi i m(B_{s_2} - B_{s_1})\right)\right] = \exp\left(-2\pi^2 |m|^2 |s_2 - s_1|^{2H}\right).$$

Upper bound

We find

$$\mathbb{E}\left[|\hat{\mu}_T(m)|^2\right] = \int_0^T \int_0^T \exp\left(-2\pi^2|m|^2|s_2 - s_1|^{2H}\right) ds_1 ds_2 \sim \frac{T}{|m|^{\frac{1}{H}}}.$$

Thus

$$\mathbb{E}\left[\int_{\mathbb{T}^d} |\nabla\Delta^{-1}P_\varepsilon(\mu_T - 1)|^2\right] \lesssim \sum_{m\in\mathbb{Z}^d\setminus\{0\}} e^{-\varepsilon|m|^2} \frac{T}{|m|^{2+1/H}}$$

$$\lesssim T \cdot \begin{cases} & \text{for } d < \frac{1}{H} + 2 \\ |\log(\varepsilon)| & \text{for } d = \frac{1}{H} + 2 \\ 1/\sqrt{\varepsilon}^{d-2-1/H} & \text{for } d > \frac{1}{H} + 2. \end{cases}$$

Taking expectation in the upper bound

$$\mathbb{E}\left[W_1(\mu_T, T\mathcal{L}^d_{\mathbb{T}^d})\right] \le \inf_{\varepsilon>0} C\sqrt{\varepsilon}T + \mathbb{E}\left[\int_{\mathbb{T}^d} |\nabla\Delta^{-1}P_\varepsilon(\mu_T - T)|^2\right]^{1/2}$$

We conclude choosing

$$\sqrt{\varepsilon} = \begin{cases} 0 & \text{for } d < \frac{1}{H} + 2 \\ T^{-1/2} & \text{for } d = \frac{1}{H} + 2 \\ T^{-\frac{1}{2-1/H}} & \text{for } d > \frac{1}{H} + 2. \end{cases}$$

Lower bound

Taking expectation in the lower bound we have

$$\mathbb{E}\left[W_1(\mu_T, T\mathcal{L}^d_{\mathbb{T}^d})\right]$$
$$\geq \sup_{M,\varepsilon>0}\left\{\frac{1}{M}\mathbb{E}\left[\int_{\mathbb{T}^d}|\nabla\Delta^{-1}P_\varepsilon(\mu_T - T)|^2\right] - \frac{C}{M^3}\mathbb{E}\left[\int_{\mathbb{T}^d}|\nabla\Delta^{-1}P_\varepsilon(\mu_T - T)|^4\right]\right\},$$

Assume that for some constant $C > 0$, we have a reverse Hölder inequality

$$\mathbb{E}\left[\int_{\mathbb{T}^d}|\nabla\Delta^{-1}P_\varepsilon(\mu_T - T)|^4\right] \leq C\mathbb{E}\left[\int_{\mathbb{T}^d}|\nabla\Delta^{-1}P_\varepsilon(\mu_T - T)|^2\right]^2$$

then choosing

$$M = K\mathbb{E}\left[\int_{\mathbb{T}^d}|\nabla\Delta^{-1}P_\varepsilon(\mu_T - T)|^2\right]^{1/2}$$

for some constant $K > 0$ (large but fixed) we have

$$\mathbb{E}\left[W_1(\mu_T, T\mathcal{L}^d_{\mathbb{T}^d})\right] \geq \mathbb{E}\left[\int_{\mathbb{T}^d}|\nabla\Delta^{-1}P_\varepsilon(\mu_T - T)|^2\right]^{1/2}\left(\frac{1}{K} - \frac{C}{K^3}\right).$$

We conclude e.g. if $\varepsilon = \varepsilon(T)$ is chosen as in the upper bound.

Fourth moment estimates

To prove

$$\mathbb{E}\left[\int_{\mathbb{T}^d} |\nabla\Delta^{-1}P_\varepsilon(\mu_T - T)|^4\right] \leq C\mathbb{E}\left[\int_{\mathbb{T}^d} |\nabla\Delta^{-1}P_\varepsilon(\mu_T - T)|^2\right]^2$$

we use lower bounds on the spectrum of the 3×3 covariance matrix for increments of fBM.
Cases:

1. $H = 1/2$: the matrix is diagonal
2. $H < 1/2$: diagonally dominant
3. $H > 1/2$: we use represent fBM as stochastic integral. Bounds look like

$$\mathbb{E}\left[\prod_{i=1}^{4} \hat{\mu}_T(m_i)\right] \lesssim T \sum_{\{i,j,k\}\subseteq\{1,2,3,4\}} (|m_i||m_i + m_j||m_i + m_j + m_k|)^{-1/H}$$

and then we use Young convolution inequality to bound the resulting series.
Is there a simpler proof?

Also F.Y. Wang needs to estimate higher order moments: his proof uses Markov property and heat kernel estimates (still very demanding).

A possible strategy via Itô trick

For $H = 1/2$ (and assuming stationarity) we can use the Itô trick to estimate

$$\mathbb{E}\left[\left|\int_0^T \nabla\Delta^{-1} P_\varepsilon(\delta_0 - 1)(B_s)\right|^p\right]$$

Write Itô formula

$$f(B_T) - f(B_0) - \int_0^T \nabla f(B_s) dB_s = \frac{1}{2}\int_0^T \Delta f(B_s) ds.$$

Set $g = \Delta f$ (g must have zero integral on $\mathbb{T}^d$) so that

$$\int g \, \mathrm{d}\mu_T = 2\left(\Delta^{-1} g(B_T) - \Delta^{-1} g(B_0) - \int_0^T \nabla\Delta^{-1} g(B_s) dB_s\right).$$

Take $p = 2^k$, use BDG inequality

$$\mathbb{E}\left[\left|\int g \, \mathrm{d}\mu_T\right|^p\right] \lesssim \mathbb{E}\left[|\Delta^{-1} g(B_0)|^p\right] + \mathbb{E}\left[\left|\int_0^T \left|\nabla\Delta^{-1} g\right|^2 (B_s) ds\right|^{p/2}\right],$$

and iterate (but each time subtract the spatial average). For $p = 2$ it gives

$$\mathbb{E}\left[\left|\int g \, \mathrm{d}\mu_T\right|^2\right] \lesssim \mathbb{E}\left[|\Delta^{-1} g(B_0)|^2\right] + T\mathbb{E}\left[\left|\nabla\Delta^{-1} g\right|^2 (B_0)\right].$$

Conclusion

1. We proved upper and lower bounds for the W_1 distance between a fractional Brownian path and the uniform measure (on $\mathbb{T}^d$).
   A transition between different rates occurs at
   $$d = 2 + \frac{1}{H}.$$
2. Many open problems, e.g., existence of limits (possibly for W_2 instead)
3. Connections with regularization by noise or covering times should be better investigated.