$\boxed{\text{Phase transitions for } \phi^4_3}$

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Based on joint work with

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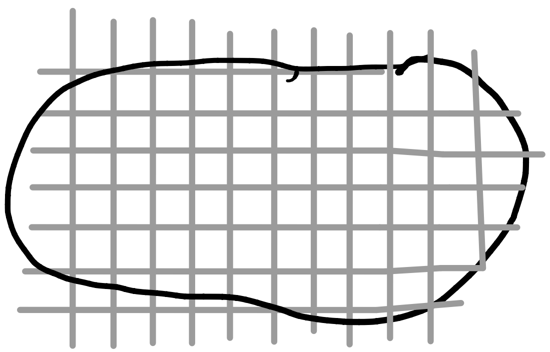
ϕ^4 Theory

Formally: Probability distribution on "functions" over set $\Lambda \subseteq \mathbb{R}^d$

$$\boxed{\sim \exp\left(-\int_\Lambda \frac{|\nabla\phi|^2(x)}{2} + |\phi|^4(x) \quad dx\right)\prod_{x\in\Lambda} d\phi(x)}$$

Doesn't make sense as it stands!!

Interpretation: • take discretisation



$$\sim \exp\left(-\sum_{x\in\Lambda_\varepsilon}\varepsilon^d\,|\nabla_\varepsilon\phi(x)|^2 - \sum_{x\in\Lambda_\varepsilon}\varepsilon^d\,|\phi(x)|^4\right)\prod_{x\in\Lambda_\varepsilon} d\phi(x)$$

$\Lambda_\varepsilon = \Lambda \cap \varepsilon\mathbb{Z}^d$

• take limit $\varepsilon \downarrow 0$ (ultraviolet).

(Renormalisation ...)

ϕ^4 measure and SPDEs

Over $\mathbb{R}^d$: measure $\sim \exp\left(-V(x)\,dx\right)$ reversible for

$$\boxed{dx_t = -\nabla V(x)\,dt + \sqrt{2}\,dw_t}$$

Langevin dynamic

Infinite dimensional analogue:

$$\partial_t \phi_t = \underbrace{\left(\Delta\phi - \phi^3\right)}_{\text{"}-\nabla\mathcal{H}\text{"}} \quad \underbrace{+\sqrt{2}\,\xi}_{\text{"}dW_t\text{"}}$$

space-time white noise.

SPDEs of this type studied recently:

- Hairer (regularity structures) 2014,...
- Gubinelli et al (paracontrolled distributions) 2015.

Construction of measures

Gaussian case (no nonlinearity) $\to$ always possible.

$$\boxed{\sim \exp\left(-\int |\nabla\phi|^2(x)\, dx\right) \prod_x d\phi(x)}$$

so called *Gaussian free field*.

E.g. on $\mathbb{T}^d$, representation as

"iid" Gaussian

$$\boxed{\sum_{k\in\mathbb{Z}^d} \frac{X_k}{|k|} e^{ik.x}}$$

(modulo 0-th Fourier mode)

Converges in space of low regularity $\boxed{< \frac{2-d}{2}}$.

Non-Gaussian case

$$\boxed{\sim \exp\left(-\int_\Lambda \frac{|\nabla\phi|^2(x)}{2} + \phi^4(x) \quad dx\right)\prod_{x\in\Lambda} d\phi(x)}$$

$\boxed{d=1}$: OK, define through density w.r.t. Gaussian.

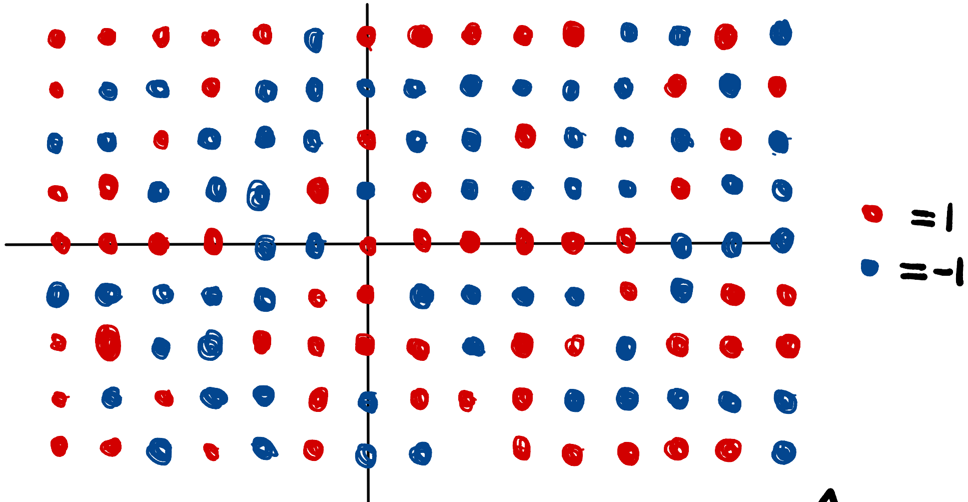
$\boxed{d\geq 2}$ Low regularity, need for renormalisation.
(Nelson '66)

$\boxed{d=3}$ Non-linear measure singular w.r.t. GFF
(Barashkov - Gubinelli 2019)

$\boxed{d\geq 4}$ Triviality. $d>4$ Aizenman '82, Fröhlich '82.
$d=4$ Aizenman - Duminil-Copin 2019.

ϕ^4 as continuous Ising model:

Ising model



$\Lambda \subset\subset \mathbb{Z}^d$; configurations in $\{\pm 1\}^\Lambda$

Hamiltonian: $\mathcal{H}(\sigma) = -\frac{1}{2}\sum_{i,j} K_{ij}\,\sigma_i\sigma_j$

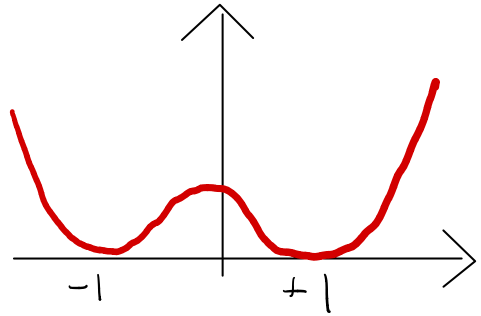
interaction ≥ 0

Gibbs measure: $\mu_\beta(\sigma) = \frac{1}{Z}\exp\left(-\beta\,\mathcal{H}(\sigma)\right)$

"inverse temperature"

- Continuous spin

$$\delta_{-1} + \delta_{+1} \rightsquigarrow \exp\left(-\lambda\,(\phi^2-1)^2\right)d\phi$$



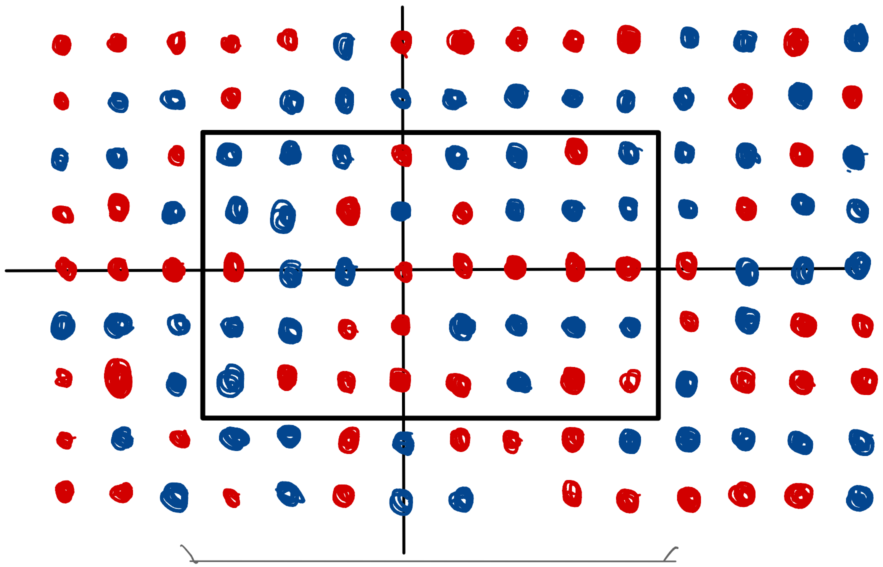
- interaction:

$$-\frac{1}{2}\sigma_i\sigma_j \rightsquigarrow \frac{1}{2}\left(\phi_i - \phi_j\right)^2$$

ϕ^4 as scaling limit

Classical: Griffith - Simon '73.

Kac - Interaction:



γ^{-1}

$$K_{ij} \sim \gamma^d \mathcal{K}(\gamma(i-j))$$

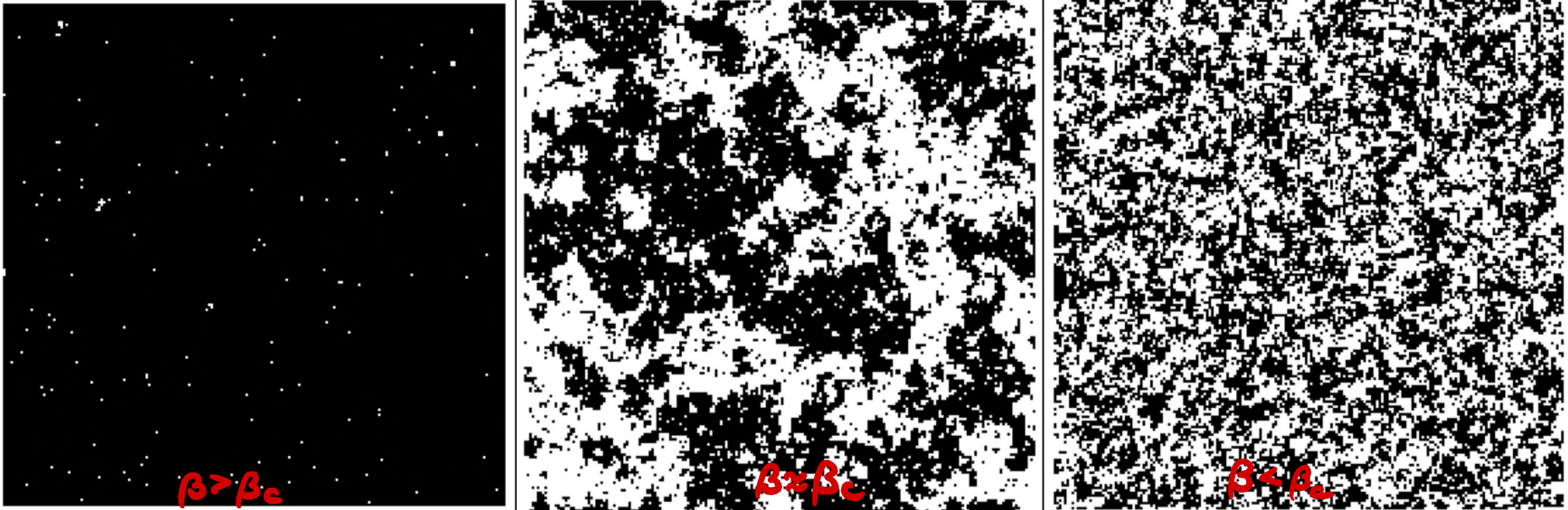
local mean field.

- Near critical temperature: $\beta = 1 + (\text{correction})$
- On correct scales: $\hat{x} = \gamma^{\frac{4}{4-d}} x$

$$\rightsquigarrow \quad \boxed{\gamma^{-\frac{d}{4-d}} \sigma \quad \Rightarrow \quad \phi^4}$$

$d=2$ Cassandro, Marra, Presutti 95

Hairer, Iberti 2018

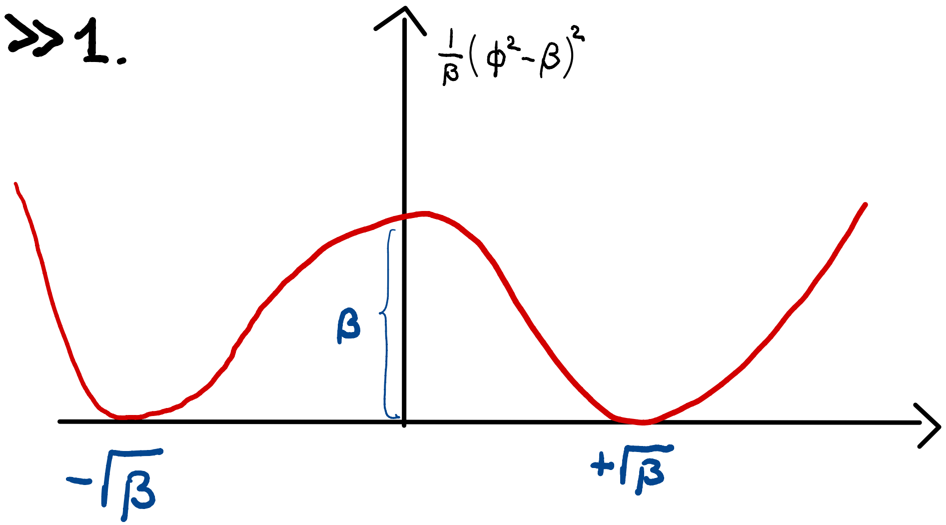


$\beta > \beta_c$
$\beta \approx \beta_c$
$\beta < \beta_c$

Low temperature ϕ^4

$$\boxed{\nu_{\beta,N} \sim \exp\left(- \int_{N\mathbb{T}^3} \frac{|\nabla \phi|^2(x)}{2} + \frac{1}{\beta}(\phi^2 - \beta)^2 + \text{c.t.} \; dx \right) \prod_{x \in N\mathbb{T}^3} d\phi(x)} \quad (1).$$

$\beta \gg 1.$



Intuition:

transition from $\pm\sqrt{\beta}$ costs more for $|\nabla\phi|^2$.

Thm: Chandra - Gunaratnam - W. arXiv:2006.15933

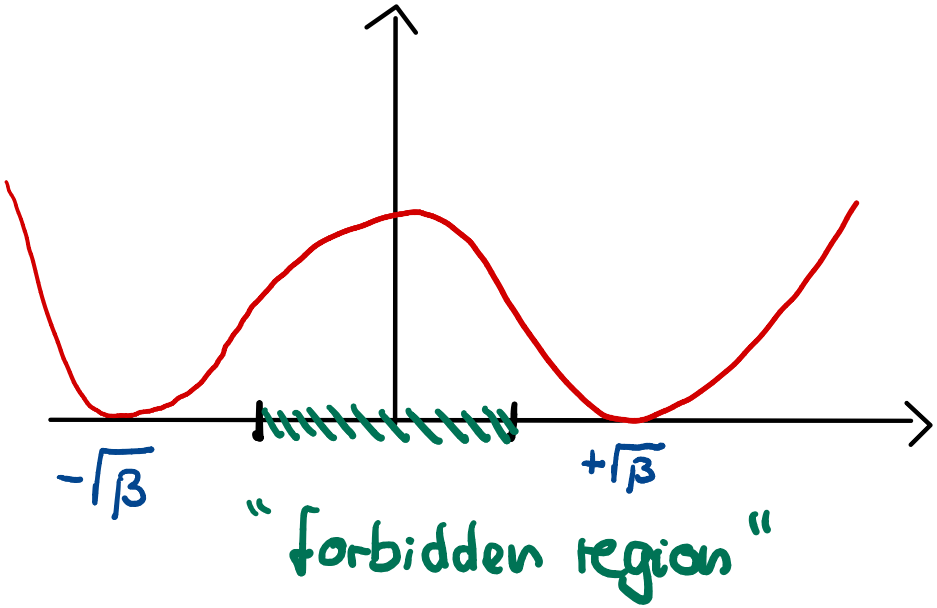
Assume $\beta \gg 1$, $N \geq 4$ dyadic

any number in $(0,1)$ is OK.

$$\boxed{\frac{1}{N^2} \log \nu_{\beta,N}\left(\left| \fint_{N\mathbb{T}^3} \phi \right| \leq \underline{\tfrac{1}{2}\sqrt{\beta}} \right) \leq -C\sqrt{\beta}}$$

"mean magnetisation"

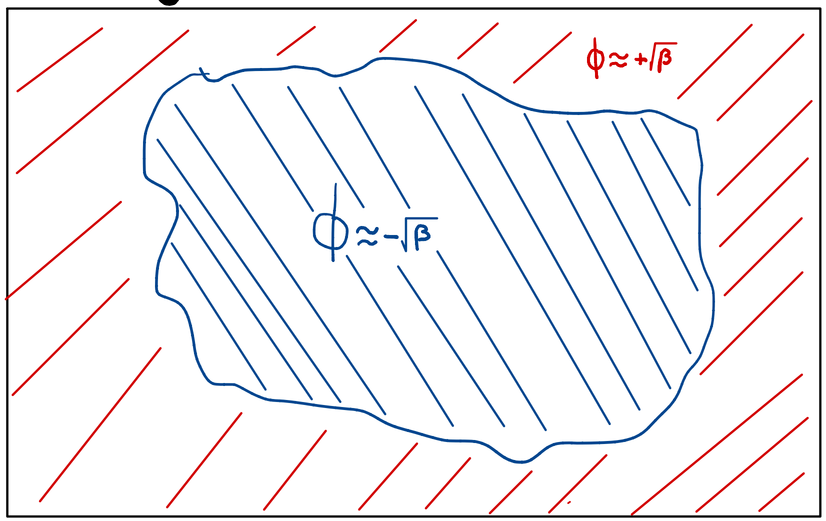
$\beta \gg 1.$



Surface order bounds

$$\boxed{\frac{1}{N^2}\log \nu_{\beta,N}\left(\left| \fint_{N\mathbb{T}^3} \phi \right| \leq \frac{1}{2}\sqrt{\beta} \right) \leq -C\sqrt{\beta}}$$

Configuration with $\fint \phi \approx 0$



Interface $\gtrsim N^2$

(by Isoperimetric inequality)

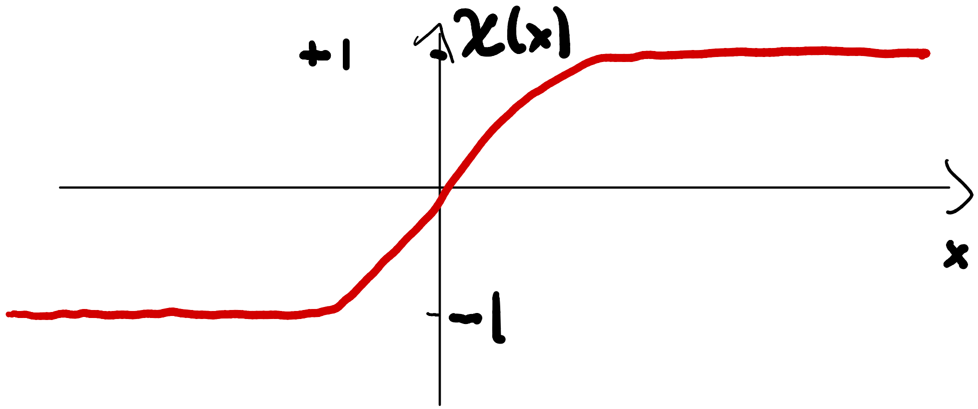
Corollary: (breakdown of ergodicity)

$\lambda_{\beta,N}$ = spectral gap of ϕ^4 SPDE (associated to $\nu_{\beta,N}$)

Then

$$\boxed{\frac{1}{N^2}\log \lambda_{\beta,N} \leq -C\sqrt{\beta}}$$

Proof: plug $\chi(f\phi)$ into Dirichlet form.



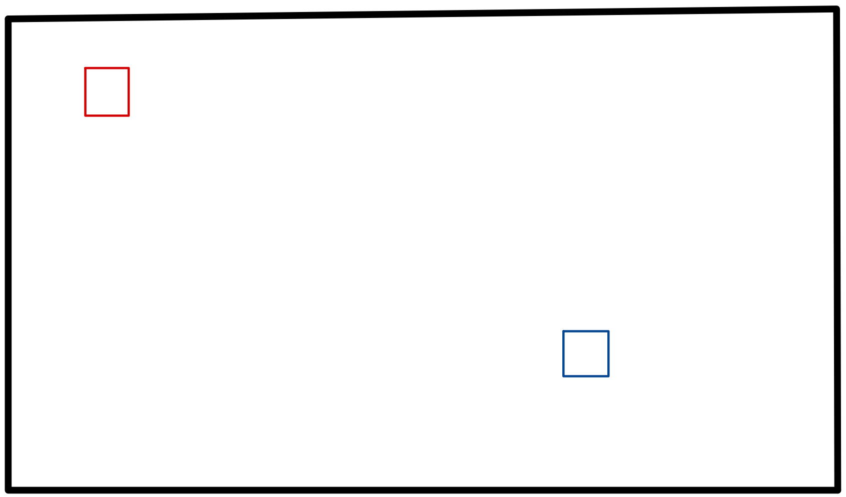
Related results:

- Glimm - Jaffe - Spencer '75

Long range order $d=2$ β large enough

$\square$ = box $(k,k+1)\times(k',k'+1)$, $\phi(\square) := \int_{\square}\phi$ average over box

$$\langle \mathbb{1}_{\phi(\square)>0}\mathbb{1}_{\phi(\square')<0}\rangle - \langle \mathbb{1}_{\phi(\square)>0}\rangle\langle \mathbb{1}_{\phi(\square')<0}\rangle \geq 1/8$$



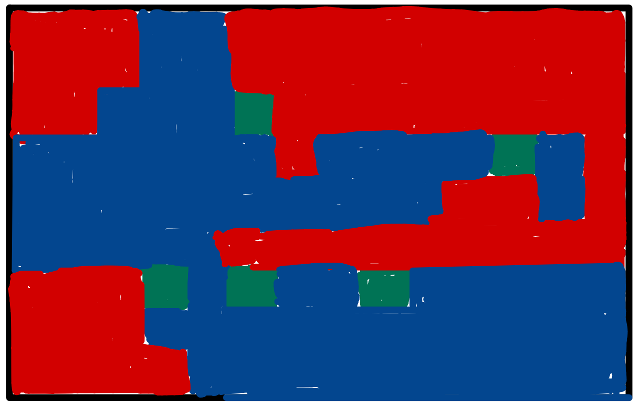
Strong interaction even if $\square$ and $\square$ far apart

Key technical step   Peierls type estimate

Def: Block $\square = (k,k+1) \times (k',k'+1)$ plus (resp. minus) valued if

$$|\phi(\square) - \sqrt{\beta}| < \sqrt{\beta}\,\delta \quad \text{small fixed} \qquad (\text{resp. } |\phi(\square) + \sqrt{\beta}| < \sqrt{\beta}\,\delta)$$

- Block $\square$ plus good if it and all $*$-neighbours plus valued.
  ... minus good ...
- $\square$ bad if not plus/minus good.



Prop: For any collection $\mathcal{B}$ of boxes

$$\nu_{\beta,N}\left(\forall \square \in \mathcal{B},\ \square \text{ bad}\right) \leq \exp\left(-C\sqrt{\beta}\,|\mathcal{B}|\right)$$

extensive in $|\mathcal{B}|$

Punishing bad boxes

$\square$ fixed box:

$$Q_1(\square) := \frac{1}{\sqrt{\beta}} \int_\square \beta - :\phi^2:$$

$$Q_2(\square) := \frac{1}{\sqrt{\beta}} \int_\square :\phi^2: - \phi(\square)^2$$

intuitively controlled by potential

$$Q_3(\square, \square') := \phi(\square) - \phi(\square')$$

controlled by Gaussian covariance

Prop: B_1, B_2, B_3 collection of boxes, β large enough

$$\left\langle \prod_{i=1}^{3} \cosh\left(Q_i(B_i)\right)\right\rangle \lesssim \exp\left(C\left(|B_1| + |B_2| + |B_3|\right)\right)$$

extensive in B_i

Implies Peierls bound

Calculating generalised partition functions

$$Z' := \int \exp(Q_i(B_i)) \exp\left(\int \tfrac{1}{\beta}(\phi^2-\beta)^2 + \text{c.t.}\right) \underset{\text{Gaussian}}{\mu}(d\phi)$$

- $d=2$ Nelson trick (Nelson '66)
- $d=3$ phase cell localization (Glimm-Jaffe '73)

  RG (Benfatto, Cassandro, ... '80, Brydges, Dimock, Hurd '95)

- Stochastic PDE don't help

- New method, Boué-Dupuis, Barashkov-Gubinelli Duke '20.

Thm (Boué-Dupuis '98)

Let $(B_t)_{t\in[0,1]}$ standard $\mathbb{R}^d$ (or $\mathbb{C}^d$) valued BM.

$H : C([0,1],\mathbb{R}^d) \longrightarrow \mathbb{R}$ bdd. measurable. Then

$$\boxed{-\log \mathbb{E} \exp(-\mathcal{H}(B)) = \inf_{\substack{v \text{ adapted} \\ \text{bdd}}} \mathbb{E}\left[\mathcal{H}\left(B + \int_0^{\cdot} v_t \, dt\right) + \frac{1}{2}\int_0^1 v_t^2 \, dt\right]}$$

Version of Gibbs variational principle:

$$-\log \mathbb{E}_{\mathbb{P}} \exp(-\mathcal{H}(B)) = \inf_{\substack{\mathbb{Q} \\ \text{proba measure} \\ \text{on } C([0,1],\mathbb{R}^d)}} \mathbb{E}_{\mathbb{Q}} \mathcal{H}(B) + \underbrace{R(\mathbb{Q} \,|\, \overset{\text{Wiener measure}}{\mathbb{P}})}_{\text{relative entropy}}.$$

How is this useful? / Where is the Brownian motion?

- $\mu(d\varphi)$ Gaussian measure (covariance operator $(-\Delta)^{-1} + \eta$).

Barashkov-Gubinelli introduce auxiliary time!

- Introduce function valued Brownian motion $(\varphi(t,x))_{\substack{t\in[0,1] \\ x\in\mathbb{T}}}$ s.th. law of $\varphi(1,\cdot)$ is μ.

- Can be done easily: Fourier series:

$$\varphi(t,x) = \sum_{k\in\mathbb{Z}/L} \frac{1}{L} \frac{L^{1/2} B_k(t)}{\sqrt{(\pi k)^2 + \eta}} e^{i\pi k x}$$

- Flexibility here.

Then for $\mathcal{H}$ reasonable:

$$-\log \mathbb{E}_\mu\left[\exp\left(-\mathcal{H}(\varphi)\right)\right]$$

$$= -\log \mathbb{E}_\mu\left[\exp\left(-\mathcal{H}(\varphi(1))\right)\right] \qquad \text{(by definition)}$$

$$= \boxed{\inf_V \mathbb{E}\left[\mathcal{H}\left(\varphi(1)+V(1)\right) + \frac{1}{2}\int_0^1 \int |\nabla \dot{V}|^2 + \eta V^2 \, dx \, dt\right]}$$

Barashkov - Gubinelli arXiv '18, Duke '20 Use to show UV stability of ϕ^4_3.

Main challenge for us: get β dependence right.

Conclusion:

- ϕ^4_3 theory in "low temperature" regime.
- we get "large deviation bound" for atypical magnetization. Correct surface-order scaling in L.
- Implies decay of spectral gap for dynamics.
- Method inspired by Glimm-Jaffe-Spencer - compare with lattice spin model.
- Key technical tool: Barashkov-Gubinelli variational approach to UV stability