The geometry of the space of branched rough paths

Lorenzo Zambotti (Sorbonne University, Paris 6)
joint work with Nikolas Tapia

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Rough Paths

Rough Paths were invented by Terry Lyons in the 90s in order to develop a new approach to stochastic integration, following some earlier intuitions of Hans Föllmer.

Massimiliano Gubinelli introduced controlled paths and later branched Rough Paths, which we are going to discuss.

Branched Rough Paths have been generalised by Martin Hairer with the introduction of Regularity Structures

Rough Paths and models of Regularity Structures are defined by a mixture of analytic and algebraic conditions.

Renormalisation

In Regularity Structures a key tool is the Renormalisation step.

This step is based on the knowledge of the group of automorphisms of the space of models.

Curiously, this question has been asked earlier for regularity structures than for rough paths.

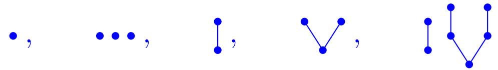
In fact, for regularity structures the answer has been given by Bruned-Hairer-Z. but only for stationary models.

We want to discuss the same problem for branched rough paths (see also Bruned-Chevyrev-Friz-Preiss).

Branched rough paths (Gubinelli)

We use (non-planar) rooted forests to code relevant stochastic integrals.

$$\tau = \quad \emptyset,$$

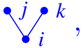


- $\langle \mathbb{X}^{\mathrm{Ito}}_{st}, \tau \rangle = 1$
- $\langle \mathbb{X}^{\mathrm{Ito}}_{st}, \tau \rangle = B_t - B_s = \int_s^t \mathrm{d}B_u$
- $\langle \mathbb{X}^{\mathrm{Ito}}_{st}, \tau \rangle = (B_t - B_s)^3$
- $\langle \mathbb{X}^{\mathrm{Ito}}_{st}, \tau \rangle = \int_s^t (B_u - B_s)\, \mathrm{d}B_u$
- $\langle \mathbb{X}^{\mathrm{Ito}}_{st}, \tau \rangle = \int_s^t (B_u - B_s)^2\, \mathrm{d}B_u$, with $(B_t)_{t \geq 0}$ a standard BM.

The vector case

If $B = (B^1, \dots, B^d)$ is d-dimensional BM, we consider rooted trees with edges decorated by $\{1, \dots, d\}$

$$\tau = \quad \emptyset, \quad \bullet\, i\,, \quad \bullet\, i \bullet j \bullet k\,,$$



- $\langle \mathbb{X}^{\mathrm{Ito}}_{st}, \tau \rangle = 1$
- $\langle \mathbb{X}^{\mathrm{Ito}}_{st}, \tau \rangle = B^i_t - B^i_s = \int_s^t \, \mathrm{d}B^i_u$
- $\langle \mathbb{X}^{\mathrm{Ito}}_{st}, \tau \rangle = (B^i_t - B^i_s)(B^j_t - B^j_s)(B^k_t - B^k_s)$
- $\langle \mathbb{X}^{\mathrm{Ito}}_{st}, \tau \rangle = \int_s^t (B^j_u - B^j_s) \, \mathrm{d}B^i_u$
- $\langle \mathbb{X}^{\mathrm{Ito}}_{st}, \tau \rangle = \int_s^t (B^j_u - B^j_s)(B^k_u - B^k_s) \, \mathrm{d}B^i_u$

The Connes-Kremier Hopf algebra

We call H the vector space generated by such decorated rooted trees. We endow H with two operations:

- a commutative multiplication $\mathcal{M} : H \otimes H \to H$, linearly generated by

$$\mathcal{M}(\tau \otimes \sigma) := \tau \sqcup \sigma$$

  with neutral element given by $\mathbf{1} := \emptyset$

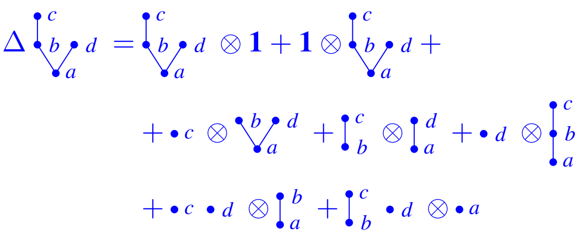
- a comultiplication or coproduct $\Delta : H \to H \otimes H$, linearly generated by

$$\Delta\tau = \tau \otimes \mathbf{1} + \mathbf{1} \otimes \tau + \sum_{C \in \mathfrak{A}(\tau)} P^C(\tau) \otimes R^C(\tau).$$

  where $\mathfrak{A}(\tau)$ is the set of admissible cuts of τ.

This is the Butcher-Connes-Kreimer Hopf algebra, well known in renormalisation and in geometric numerical analysis.

An example



but we do not have the terms

A recursive formula

H has a recursive structure: all elements of H are obtained from $\mathbf{1}$ with a finite number of products and of applications of the operators

$$\tau \to [\tau]_i$$

where we join all roots in to a new root $\bullet\, i$.

The coproduct Δ has the recursive construction

$$\Delta\mathbf{1} = \mathbf{1} \otimes \mathbf{1}, \qquad \Delta(\tau_1 \cdots \tau_n) = (\Delta\tau_1) \cdots (\Delta\tau_n)$$

$$\Delta[\tau]_i = [\tau]_i \otimes \mathbf{1} + (\mathrm{id} \otimes [\cdot]_i)\Delta\tau.$$

Motivation

This allows to define $\mathbb{X}^{\mathrm{Ito}}_{st} : H \to \mathbb{R}$ recursively as a linear functional:

1. $\langle \mathbb{X}^{\mathrm{Ito}}_{st}, \mathbf{1} \rangle = 1$
2. $\langle \mathbb{X}^{\mathrm{Ito}}_{st}, \tau_1 \cdots \tau_n \rangle = \langle \mathbb{X}^{\mathrm{Ito}}_{st}, \tau_1 \rangle \cdots \langle \mathbb{X}^{\mathrm{Ito}}_{st}, \tau_n \rangle$
3. $$\langle \mathbb{X}^{\mathrm{Ito}}_{st}, [\tau]_i \rangle = \int_s^t \langle \mathbb{X}^{\mathrm{Ito}}_{sr}, \tau \rangle \, \mathrm{d}B^i_r$$

In fact this definition holds in the Stratonovich case or for a smooth X or for a semimartingale Y, if in the third condition $\mathrm{d}B^i_r$ is replaced by, respectively,

$$\circ \, \mathrm{d}B^i_r, \qquad \dot{X}^i_r \, \mathrm{d}r, \qquad \mathrm{d}Y^i_r.$$

The Chen relation

Theorem
For all $s \le u \le t$:

$$\langle \mathbb{X}_{st}, \tau \rangle = \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ut}, \Delta\tau \rangle, \qquad \forall \tau \in H.$$

$$\begin{aligned}
\langle \mathbb{X}_{st}, [\tau]_i \rangle &= \int_s^t (\mathbb{X}_{sr}\tau)\,\mathrm{d}B^i_r = \int_s^u (\mathbb{X}_{sr}\tau)\,\mathrm{d}B^i_r + \int_u^t (\mathbb{X}_{sr}\tau)\,\mathrm{d}B^i_r \\
&= \langle \mathbb{X}_{su}, [\tau]_i \rangle + \int_u^t \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ur}, \Delta\tau \rangle\,\mathrm{d}B^i_r \\
&= \langle \mathbb{X}_{su}, [\tau]_i \rangle + \langle \mathbb{X}_{su} \otimes \int_u^t \mathbb{X}_{ur}\,\mathrm{d}B^i_r, \Delta\tau \rangle \\
&= \langle \mathbb{X}_{su}, [\tau]_i \rangle + \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ut}[\cdot]_i, \Delta\tau \rangle \\
&= \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ut}, [\tau]_i \otimes 1 + (\mathrm{id} \otimes [\cdot]_i)\Delta\tau \rangle = \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ut}, \Delta[\tau]_i \rangle.
\end{aligned}$$

This also holds for the other possible integrals: $\mathrm{d}B^i_r \to \circ\,\mathrm{d}B^i_r,\ \dot{X}^i_r\,\mathrm{d}r,\ \mathrm{d}Y^i_r$.

Branched rough paths

In 2006 Massimiliano defines a branched rough path of regularity $\gamma > 0$ as a function $\mathbb{X} : [0,T]^2 \to H^*$ s.t.

- $\langle \mathbb{X}_{st}, \tau \rangle = \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ut}, \Delta\tau \rangle, \quad \forall \tau \in H.$
- $\langle \mathbb{X}_{st}, \tau_1\tau_2 \rangle = \langle \mathbb{X}_{st}, \tau_1 \rangle \langle \mathbb{X}_{st}, \tau_2 \rangle.$
- $\sup_{s \neq t} [|\langle \mathbb{X}_{st}, \tau \rangle| / |t-s|^{\gamma|\tau|}] < +\infty$, for all $\tau \in H$, where

$$|\tau| := \#\{\text{nodes of } \tau\}.$$

Notations and presentation follow [Hairer-Kelly 2014].

Note the non-linear constraints.

The first condition is a lift to H^* of the concatenation of paths:

$$(X_r)_{r\in[s,t]} = (X_r)_{r\in[s,u]} \star (X_r)_{r\in[u,t]}.$$

The extension Theorem

Theorem (T. Lyons, M. Gubinelli)

Given a branched rough path $\mathbb{X}$ *of regularity* $\gamma > 0$*, the values* $(\mathbb{X}\tau, |\tau| = m, m > N)$ *are uniquely determined by the values of* $(\mathbb{X}\tau, |\tau| = m, m \le N)$*, where* $N := \lfloor 1/\gamma \rfloor$.

Proof.

By recurrence on $m \ge N + 1$. We have for all τ such that $|\tau| = m$

$$(\delta_2 \mathbb{X}\tau)_{sut} = (\mathbb{X}_{su} \otimes \mathbb{X}_{ut})\Delta'\tau$$

where $\Delta'\tau := \Delta\tau - \mathbf{1} \otimes \tau - \tau \otimes \mathbf{1}$ is the reduced coproduct. We need now the Sewing Lemma... $\square$

A cochain complex

Les us define for $n \geq 1$ the set C_n of continuous functions $f : [0,T]^n \to \mathbb{R}$ such that

$$f_{t_1 \cdots t_n} = 0 \qquad \text{if} \qquad t_i = t_{i+1} \qquad \text{for some} \qquad 1 \leq i \leq n-1,$$

$$\delta_n : C_n \to C_{n+1}, \qquad (\delta_n f)_{t_1 \ldots t_{n+1}} = \sum_{k=1}^{n+1} (-1)^{n+2-k} f_{t_1 \ldots \cancel{t_k} \ldots t_{n+1}}.$$

For instance $\delta_1 g_{st} = g_t - g_s$, $\delta_2 f_{sut} = f_{st} - f_{su} - f_{ut}$. Then

- $\delta_{n+1} \circ \delta_n \equiv 0$ (exercise!)
- if $g \in C_{n+1}$ and $\delta_{n+1}\, g = 0$, then $g = \delta_n f$ with $f \in C_n$ (exercise!).

In particular we have an exact cochain complex

$$\mathbb{R} \to C_1 \xrightarrow{\delta_1} C_2 \xrightarrow{\delta_2} C_3 \xrightarrow{\delta_3} \cdots$$

The sewing lemma

For $\gamma > 0$ and $h \in C_n$ we set

$$\|h\|_\gamma := \sup_{(t_1,\dots,t_n)\in[0,T]^n} \frac{|h(t_1,\dots,t_n)|}{|t_n - t_1|^\gamma}$$

and we say that $h \in C_n^\gamma$ if $\|h\|_\gamma < +\infty$. We also set $C_n^{\gamma+} := \cup_{\beta>\gamma} C_n^\beta$.

Theorem (Gubinelli)

There exists a unique map $\Lambda : C_3^{1+} \cap \delta_2 C_2 \to C_2^{1+}$ *such that* $\delta_2 \Lambda = \mathrm{id}_{C_3^{1+}\cap\,\delta_2 C_2}$. *Moreover* Λ *satifies for all* $\gamma > 1$

$$\|\Lambda B\|_\gamma \le K_\gamma \|B\|_\gamma, \qquad B \in C_3^{1+} \cap \delta_2 C_2.$$

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By recurrence on $m \geq N+1$. We have for all τ such that $|\tau| = m$

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where $\Delta'\tau := \Delta\tau - \mathbf{1} \otimes \tau - \tau \otimes \mathbf{1}$ is the reduced coproduct. We conclude by recurrence and by the Sewing Lemma since we obtain $(\delta_2 \mathbb{X}\tau) \in C_3^{m\gamma}$ with $m\gamma > 1$. □

It is enough to control only finitely many objects.

Trees with low degree

The first question is the following.

Let $|\tau| \leq N$. Chen's relation states that

$$(\delta_2 \mathbb{X}\tau)_{sut} = (\mathbb{X}_{su} \otimes \mathbb{X}_{ut})\Delta'\tau. \tag{1}$$

The Sewing Lemma does not apply here, since $|\tau|\gamma \leq 1$.

Indeed, the RHS of (1) does not uniquely determine the LHS: if $g^\tau \in C^{\gamma|\tau|}([0,T])$ and we set

$$(g\mathbb{X})_{st}\tau := \mathbb{X}_{st}\tau + g_t^\tau - g_s^\tau,$$

then

$$(\delta_2(g\mathbb{X})\tau)_{sut} = (\mathbb{X}_{su} \otimes \mathbb{X}_{ut})\Delta'\tau, \qquad |(g\mathbb{X})_{st}\tau| \lesssim |t-s|^{\gamma|\tau|}.$$

Trees with low degree

However this raises a second question.

Let $|\tau| \leq N-1$. If we define $(g\mathbb{X})\tau$ as above, how does this propagate to higher levels (but still $\leq N$ so that the Sewing Lemma does not apply).

In other words, if $|\tau|+1 \leq |\sigma| \leq N$, how is $(g\mathbb{X})\sigma$ defined? Chen's relation states that

$$(\delta_2 \, (g\mathbb{X})\sigma)_{sut} = (g\mathbb{X}_{su} \otimes g\mathbb{X}_{ut})\Delta'\sigma.$$

The Sewing Lemma does not apply here, since $|\sigma|\gamma \leq 1$.

An example

If $\tau = \bullet\, i$, i.e. $|\tau| = 1$, then we can simply set for $g^\tau \in C^\gamma([0,1])$

$$\langle (gX)_{st}, \tau \rangle := x^i_t - x^i_s + g^i_t - g^i_s,$$

where $x^i_u := \langle X_{0u}, \bullet\, i \rangle$.
Now for $\tau = \overset{\bullet\, j}{\underset{\bullet\, i}{|}}$

$$\langle (gX)_{st}, \tau \rangle := \int_s^t (x^j_u - x^j_s + g^j_u - g^j_s)\, \mathrm{d}(x^i_u + g^i_u),$$

but this is well-defined in the Young sense only if $\gamma > 1/2$.

Once again, the problem is at the same time of analytic and algebraic nature.

The main result

Let BRP^γ be the space of γ-branched rough paths and

$$\mathcal{C}^\gamma := \{(g^\tau)_{\tau\in\mathcal{T}_N} : g^\tau \in C^{\gamma|\tau|}([0,T])\},$$

where $\mathcal{T}_N := \{\tau \text{ tree}, \ |\tau| \le N\}$. This set is a group under pointwise addition.

Then there exists a transitive free action of $\mathcal{C}^\gamma$ on BRP^γ

$$\mathcal{C}^\gamma \times \mathrm{BRP}^\gamma \ni (g, \mathbb{X}) \mapsto g\mathbb{X} \in \mathrm{BRP}^\gamma,$$

i.e.

- $g_1(g_2\mathbb{X}) = (g_1 + g_2)\mathbb{X}$
- for all $\mathbb{X} \in \mathrm{BRP}^\gamma$, the map $\mathcal{C}^\gamma \ni g \mapsto gX \in \mathrm{BRP}^\gamma$ is a bijection.

Proof

We use techniques mainly from two papers: [Lyons-Victoir 07] and [Hairer-Kelly 14].

First, we construct using [Lyons-Victoir 07] geometric rough paths $\overline{\mathbb{X}}$ whose underlying paths are in $\mathbb{R}^{\mathcal{T}_N}$.

More precisely, $g\overline{\mathbb{X}}$ is a geometric rough path over $T(\mathbb{R}^{\mathcal{T}_N})$, the tensor algebra over $\mathbb{R}^{\mathcal{T}_N}$.

By [Hairer-Kelly 14], we have a Hopf-algebra morphism $\psi : H \to T(\mathbb{R}^{\mathcal{T}_N})$, such that

$$\langle \mathbb{X}, \tau \rangle = \langle \overline{\mathbb{X}}, \psi(\tau) \rangle.$$

Finally we set

$$\langle g\mathbb{X}, \tau \rangle := \langle g\overline{\mathbb{X}}, \psi(\tau) \rangle.$$

Comments

- Lyons-Victoir (and also Hairer-Kelly) use the axiom of choice, but we use their method in a completely constructive way
- even so, the map $\mathbb{X} \mapsto \overline{\mathbb{X}}$, from a branched to a geometric rough path on a larger tensor algebra, is neither unique nor canonical. There is an action for each such choice.
- recall that for all $\mathbb{X} \in \mathrm{BRP}^\gamma$, the map $\mathcal{C}^\gamma \ni g \mapsto gX \in \mathrm{BRP}^\gamma$ is a bijection. Therefore BRP^γ is isomorphic to $\mathcal{C}^\gamma$, modulo a choice of the origin.
- It is remarkable the non-linear space BRP^γ is isomorphic to the linear space $\mathcal{C}^\gamma$.
- We have classified all possible automorphisms of BRP^γ.