Rough Differential Equations Reflected in a Time-Dependent Convex Set: Theoretical, Numerical and Statistical Results

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Introduction

The model

Consider the stochastic differential equation

$$X(t) = X_0 + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dB(s) \quad (1)$$

where B is a fractional Brownian motion.

How to constrain X to stay in a closed convex set?

If B is a Brownian motion, in Itô's calculus framework:

- Invariance condition on (b, σ).
- Skorokhod reflection problem.

Objective

If B is a fractional Brownian motion, to extend the second method in the rough paths framework.

References

Existence (only) with a constant constraint set:

- Aida (2015,2016)

Existence and uniqueness with a constant constraint set:

- Besalu & al. (2012): nonnegative constraints.
- Deya & al. (2016): 1-dimensional constraint set.
- Richard & al. (2019): penalisation in dimension 1.

Existence and uniqueness with a time-dependent constraint set:

- Falkowski & Slominski (2015): H $\in]1/2, 1[$ and a time-dependent cuboïd contraint.
- Castaing, M & Raynaud de Fitte (2018): Existence for H $\in]1/3, 1[$. Uniqueness and approximation for σ constant.

Statistical inference:

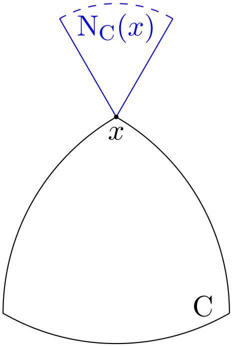
- M (2019): Nonparametric estimation of the trend.

The Skorokhod Reflection Problem

Preliminaries: normal cone

For every $x \in \mathbb{R}^e$,

$$\mathrm{N}_{\mathrm{C}}(x) := \{s : \forall y \in \mathrm{C}, \langle s, y - x \rangle \leqslant 0\}.$$



Preliminaries: the differential measure and its variation

Consider $f \in \mathrm{C}^{1\text{-var}}([0,\mathrm{T}],\mathbb{R}^e)$.

The vector measure $\mathrm{D}f$, defined on $\mathcal{B}([0,\mathrm{T}])$ by

$$\mathrm{D}f([s,t]) := f(t) - f(s),$$

is the differential measure of f.

Its variation $|\mathrm{D}f|$ is the measure defined on $\mathcal{B}([0,\mathrm{T}])$ by

$$|\mathrm{D}f|(\mathrm{A}) := \sup\{\sum_{i=1}^{n} \|\mathrm{D}f(\mathrm{B}_i)\| \; ; \; n \in \mathbb{N}^*, \\ \mathrm{B}_1,\dots,\mathrm{B}_n \in \mathcal{B}([0,\mathrm{T}]) \text{ pairwise disjoint s.t. } \mathrm{B}_i \subset \mathrm{A}\}.$$

Preliminaries: the differential measure and its variation

In the sequel,

$$\dot{f} := \frac{\mathrm{d}\mathrm{D}f}{\mathrm{d}|\mathrm{D}f|}.$$

Example. If the map f is absolutely continuous, then

$$f(t) = \int_0^t \dot{f}(s)\mathrm{d}s.$$

The Moreau sweeping process

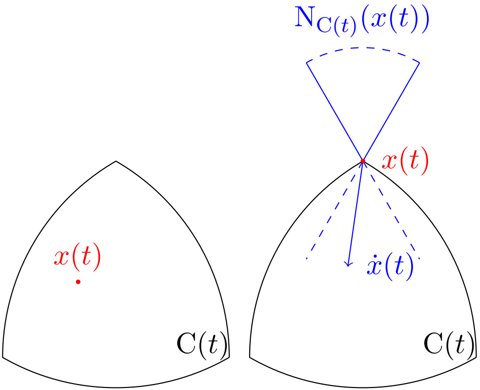
is defined by the differential inclusion

$$\begin{cases} -\dot{x}(t) \in \mathrm{N}_{\mathrm{C}(t)}(x(t)) \ |\mathrm{D}x|\text{-a.e.} \\ x(0) = x_0 \in \mathrm{C}(0) \end{cases} \tag{2}$$

where $\mathrm{C} : [0, \mathrm{T}] \rightrightarrows \mathbb{R}^e$ is a convex compact valued multifunction continuous for the Hausdorff distance.

The Moreau sweeping process: some pictures

Two situations:



The Monteiro-Marques theorem

If there exist $a \in \mathbb{R}^e$ and $r > 0$ such that

$$\overline{\mathrm{B}}_e(a, r) \subset \mathrm{C}(t) \; ; \forall t \in [0, \mathrm{T}],$$

then (2) has a unique continuous solution of finite 1-variation x such that

$$\|x\|_{1\text{-var},\mathrm{T}} \leqslant \frac{1}{2r}\|x_0 - a\|^2 \qquad (3)$$

The rough perturbed sweeping process

is defined by

$$X(t) = \underbrace{\int_0^t b(X(u))du + \int_0^t \sigma(X(u))dB(u)}_{H(t)} + Y(t) \qquad (4)$$

where

$$\begin{cases} -\dot{Y}(t) \in N_{C(t)-H(t)}(Y(t)) \ |DY|\text{-a.e.} \\ Y(0) = X_0 \end{cases} \qquad (5)$$

Existence and uniqueness of the solution

A continuity theorem
(Castaing et al. (2014))

Consider a continuous function $h : [0, \mathrm{T}] \to \mathbb{R}^e$ and

$$\begin{cases} v_h(t) = h(t) + w_h(t) \\ -\dot{w}_h(t) \in \mathrm{N}_{\mathrm{C}(t)-h(t)}(w_h(t)) \ |\mathrm{D}w_h|\text{-a.e.} \\ w_h(0) = x_0 \in \mathrm{C}(0) \end{cases} \tag{6}$$

For every sequence of continuous functions $(h_n)_{n\in\mathbb{N}}$, if

$$h_n \to h \text{ uniformly,}$$

then

$$(v_{h_n}, w_{h_n}) \to (v_h, w_h) \text{ uniformly.}$$

Existence of solutions to (4)-(5)

If there exists a continuous function $\gamma : [0, \mathrm{T}] \to \mathbb{R}^e$ such that

$$\overline{\mathrm{B}}_e(\gamma(t), r) \subset \mathrm{C}(t) \ ; \forall t \in [0, \mathrm{T}],$$

then (4)-(5) has at least one solution belonging to

$$\bigcap_{p > 1/\mathrm{H}} \mathrm{C}^{p\text{-var}}([0, \mathrm{T}], \mathbb{R}^e).$$

Existence of solutions to (4)-(5): sketch of proof

Consider the Picard scheme

$$\begin{cases} \mathrm{X}_n(t) &= \mathrm{H}_n(t) + \mathrm{Y}_n(t) \\ \mathrm{H}_n(t) &= \int_0^t b\left(\mathrm{X}_{n-1}(u)\right) \mathrm{d}u + \int_0^t \sigma\left(\mathrm{X}_{n-1}(u)\right) \mathrm{dB}(u) \\ -\dot{\mathrm{Y}}_n(t) &\in \mathrm{N}_{\mathrm{C}(t)-\mathrm{H}_n(t)}\left(\mathrm{Y}_n(t)\right) \ |\mathrm{DY}_n|\text{-a.e. with } \mathrm{Y}_n(0) = \mathrm{X}_0 \end{cases}.$$

A compactness argument using (3) and the previous continuity theorem give that $(\mathrm{X}_n, \mathrm{Y}_n)_{n\in\mathbb{N}}$ has a converging subsequence.

Its limit is a solution to (4)-(5).

Uniqueness of the solution to (4)-(5) with σ constant

Assume that σ is constant. Then, (4)-(5) is equivalent to

$$\begin{cases} X(t) = H(t) + Y(t) \\ H(t) = \int_0^t b(X(s))ds + \sigma B(t) \\ -\dot{Y}(t) \in N_{C(t)-H(t)}(Y(t)) \ |DY|\text{-a.e. with } Y(0) = X_0 \end{cases} \tag{7}$$

If there exists a continuous function $\gamma : [0, T] \to \mathbb{R}^e$ such that

$$\overline{B}_e(\gamma(t), r) \subset C(t) \ ; \forall t \in [0, T],$$

then (4)-(5) has a unique solution belonging to

$$\bigcap_{p>1/H} C^{p\text{-var}}([0, T], \mathbb{R}^e).$$

Uniqueness of the solution to (7): sketch of proof

Consider two solutions (X, Y) and $(\mathrm{X}^*, \mathrm{Y}^*)$ to (7).

Steps:

1. There exists a control function ω such that:

$$\|\mathrm{X}(t) - \mathrm{X}^*(t)\|^2 \leqslant \omega(0,t)\|\mathrm{X} - \mathrm{X}^*\|^2_{\infty,t} + \mathfrak{c}\int_0^t \langle \mathrm{X}(u) - \mathrm{X}^*(u), \mathrm{d}(\mathrm{Y} - \mathrm{Y}^*)(u)\rangle.$$

2. By the monotonicity of $\mathrm{N}_{\mathrm{C}(u)}(.)$; $u \in [0, \mathrm{T}]$,

$$\int_0^t \langle \mathrm{X}(u) - \mathrm{X}^*(u), \mathrm{d}(\mathrm{Y} - \mathrm{Y}^*)(u)\rangle \leqslant 0.$$

3. By dissecting $[0, \mathrm{T}]$ in small subintervals, $(\mathrm{X}, \mathrm{Y}) = (\mathrm{X}^*, \mathrm{Y}^*)$.

Uniqueness of the solution to (4)-(5): NSC

Consider two solutions (X, Y) and $(\mathrm{X}^*, \mathrm{Y}^*)$ to (4)-(5).

The monotonicity of the normal cone is not sufficient to prove that

$$\|\mathrm{X} - \mathrm{X}^*\|_{p\text{-var},t} = 0.$$

It is true if and only if

$$\int_s^t \langle \mathrm{R}_{\mathrm{X}}(s,u) - \mathrm{R}_{\mathrm{X}^*}(s,u), \mathrm{d}(\mathrm{Y} - \mathrm{Y}^*)(u)\rangle \leqslant 0 \qquad (8)$$

Numerical scheme

Approximation of the solution to (7)

For $n \in \mathbb{N}^*$, consider the approximation scheme

$$\begin{cases} \mathrm{X}_0^n = x_0 \\ \mathrm{X}_{k+1}^n = p^{\perp}_{\mathrm{C}(t_{k+1}^n)}(\mathrm{X}_k^n + \mathrm{H}^n(t_{k+1}^n) - \mathrm{H}^n(t_k^n)) \end{cases} \quad (9)$$

where

$$\mathrm{H}^n(t) - \mathrm{H}^n(t_k^n) = b(\mathrm{X}_k^n)(t - t_k^n) + \sigma(\mathrm{B}(t) - \mathrm{B}(t_k^n)) \ ; \ t \in [t_k^n, t_{k+1}^n[\quad (10)$$

Convergence of the scheme (9)-(10)

By putting

$$X^n(t) := X^n_k \ ; t \in [t^n_k, t^n_{k+1}[,$$

if C is Hölder continuous for the Hausdorff distance and there exists a continuous function $\gamma : [0, T] \to \mathbb{R}^e$ such that

$$\overline{B}_e(\gamma(t), r) \subset C(t) \ ; \forall t \in [0, T],$$

then $(X^n)_{n\in\mathbb{N}^*}$ converges uniformly pathwise to X.

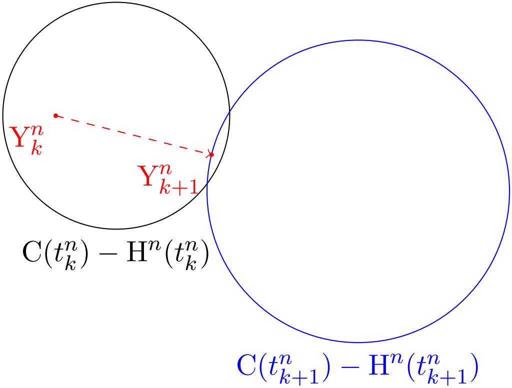
Convergence of the scheme (9)-(10): sketch of proof

Consider

$$\mathrm{Y}^n_{k+1} := \mathrm{X}^n_{k+1} - \mathrm{H}^n(t^n_{k+1}) = p^{\perp}_{\mathrm{C}(t^n_{k+1})-\mathrm{H}^n(t^n_{k+1})}(\mathrm{Y}^n_k)$$

and put

$$\mathrm{Y}^n(t) := \mathrm{Y}^n_k \ ; t \in [t^n_k, t^n_{k+1}[.$$



Convergence of the scheme (9)-(10): sketch of proof

Steps:

1. By a compactness argument, any subsequence of $(\mathrm{H}^n)_{n\in\mathbb{N}^*}$ has a subsequence converging in $\mathrm{C}^0([0,\mathrm{T}],\mathbb{R}^e)$. Its limit is denoted by H^*.
2. $(\mathrm{Y}^n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in $\mathrm{C}^0_{\text{piecewise}}([0,\mathrm{T}],\mathbb{R}^e)$.
3. The limit Y^* of $(\mathrm{Y}^n)_{n\in\mathbb{N}^*}$ satisfies

$$\langle z-\mathrm{H}^*(\tau),\mathrm{Y}^*(t)-\mathrm{Y}^*(s)\rangle \geqslant \frac{1}{2}(\|\mathrm{Y}^*(t)\|^2-\|\mathrm{Y}^*(s)\|^2) \quad (11)$$

   for every $(s,t)\in\Delta_{\mathrm{T}}$, $\tau\in[s,t]$ and $z\in\cap_{\tau\in[s,t]}\mathrm{C}(\tau)$.
4. So, $(\mathrm{X}^*,\mathrm{Y}^*)$ with $\mathrm{X}^*:=\mathrm{H}^*+\mathrm{Y}^*$ coincides with (X,Y).

Statistical inference

A nonparametric estimator of the trend: preliminaries

The solution to (7) is denoted by X_σ, $e = 1$ and $H > 1/2$.

Let us provide a converging estimator of the trend

$$\tau := x - X(0)$$

of Problem (7), where

$$\begin{cases} x(t) = h(t) + y(t) \\ h(t) = \int_0^t b(x(s))\mathrm{d}s \\ -\dot{y}(t) \in N_{C(t)-h(t)}(y(t)) \ |\mathrm{D}y|\text{-a.e. with } y(0) = X(0) \end{cases} \tag{12}$$

A nonparametric estimator of the trend

Consider

$$\widehat{\tau}_\sigma(t) := \frac{1}{h_\sigma}\int_0^t \int_0^{\mathrm{T}} \mathcal{K}\left(\frac{s-u}{h_\sigma}\right) \mathrm{d}\mathrm{X}_\sigma(s)\mathrm{d}u,$$

where $\mathcal{K}$ is a smooth kernel such that $\mathcal{K}^{-1}(\{0\})^c =]\mathrm{A},\mathrm{B}[$.

Convergence of the estimator

If C is Lipschitz continuous for the Hausdorff distance, then

$$\sup_{t\in[0,\mathrm{T}]} \mathbb{E}(|\widehat{\tau}_\sigma(t)-\tau(t)|^2) \leqslant \mathfrak{c}(\sigma^2+h_\sigma^2+\sigma^2 h_\sigma^{2\mathrm{H}-2}) \qquad (13)$$

The best possible rate is reached for $h_\sigma = \sigma^{1/(2-\mathrm{H})}$. Then,

$$\lim_{\sigma\to 0} \sigma^{\alpha-2/(2-\mathrm{H})} \sup_{t\in[0,\mathrm{T}]} \mathbb{E}(|\widehat{\tau}_\sigma(t)-\tau(t)|^2) = 0 \ ; \forall\alpha>0.$$

Convergence of the estimator: sketch of proof

On the one hand, by Slominski and Wojciechowski (2013),

$$\|\mathrm{Y}_\sigma - y\|_{\infty,\mathrm{T}} \leqslant \|\mathrm{H}_\sigma - h\|_{\infty,\mathrm{T}}.$$

Then, by Gronwall's lemma:

$$\mathbb{E}(\|\mathrm{X}_\sigma - x\|^2_{\infty,\mathrm{T}}) + \mathbb{E}(\|\mathrm{Y}_\sigma - y\|^2_{\infty,\mathrm{T}}) \leqslant \mathfrak{c}\sigma^2 \tag{14}$$

Convergence of the estimator: sketch of proof

On the other hand,

$$\widehat{\tau}_\sigma(t) - \tau(t) = \alpha_\sigma(t) + \beta_\sigma(t) + \gamma_\sigma(t) + \zeta_\sigma(t) + \eta_\sigma(t),$$

where

$$\alpha_\sigma(t) := \int_0^t \int_0^{\mathrm{T}} \mathcal{K}_{h_\sigma}(s-u)(b(\mathrm{X}_\sigma(s)) - b(x(s)))\mathrm{d}s\mathrm{d}u,$$

$$\beta_\sigma(t) := \int_0^t \int_0^{\mathrm{T}} \mathcal{K}_{h_\sigma}(s-u)b(x(s))\mathrm{d}s\mathrm{d}u - \int_0^t b(x(u))\mathrm{d}u,$$

$$\gamma_\sigma(t) := \sigma \int_0^t \int_0^{\mathrm{T}} \mathcal{K}_{h_\sigma}(s-u)\mathrm{dB}(s)\mathrm{d}u,$$

$$\zeta_\sigma(t) := \int_0^t \int_0^{\mathrm{T}} \mathcal{K}_{h_\sigma}(s-u)\mathrm{d}(\mathrm{Y}_\sigma - y)(s)\mathrm{d}u \text{ and}$$

$$\eta_\sigma(t) := \int_0^t \int_0^{\mathrm{T}} \mathcal{K}_{h_\sigma}(s-u)\mathrm{d}y(s)\mathrm{d}u - y(t) + x_0.$$

Convergence of the estimator: sketch of proof

- Inequality (14) and the Lipschitz continuity of C give

$$\sup_{t\in[0,\mathrm{T}]} \mathbb{E}(\alpha_\sigma(t)^2 + \zeta_\sigma(t)^2) \leqslant \mathfrak{c}\sigma^2 \tag{15}$$

  and

$$\sup_{t\in[0,\mathrm{T}]} \mathbb{E}(\beta_\sigma(t)^2 + \eta_\sigma(t)^2) \leqslant \mathfrak{c}h_\sigma^2 \tag{16}$$

- Memin et al. (2001) gives

$$\sup_{t\in[0,\mathrm{T}]} \mathbb{E}(\gamma_\sigma(t)^2) \leqslant \mathfrak{c}\sigma^2 h_\sigma^{2\mathrm{H}-2} \tag{17}$$

Asymptotic distribution: preliminaries

Assume that $\mathrm{C} = [l(.), u(.)]$ with $l, u \in \mathrm{C}^1([0,\mathrm{T}],\mathbb{R})$.

For any $\mathrm{I} : [0,\mathrm{T}] \rightrightarrows \mathbb{R}$ such that $\mathrm{I}(.) \subset \mathrm{C}(.)$, consider

$$\mathcal{E}_{\mathrm{I}} := \{t \in [0,\mathrm{T}] : \exists \varepsilon > 0,\ \forall s \in]t-\varepsilon, t+\varepsilon[,\ x(s) \in \mathrm{I}(s)\}.$$

Asymptotic distribution

For any $t \in \mathcal{E}_l \cup \mathcal{E}_u \cup \mathcal{E}_{\mathrm{int(C)}}$,

$$\sigma^{-1/(2-\mathrm{H})}(\widehat{\tau}_\sigma(t) - \tau(t) - \gamma_\sigma(t)) \xrightarrow[\sigma\to 0]{\mathbb{L}^2} \mu(t) \tag{18}$$

and

$$\sigma^{-1/(2-\mathrm{H})}\dot{\gamma}_\sigma(t) \xrightarrow[\sigma\to 0]{\Delta} \mathcal{N}(0, \sigma^2_{\mathrm{H},\mathcal{K}}),$$

where

- $\mu(t) := (b(x(t)) - b(x(0)) + \dot{y}(t) - \dot{y}(0)) \int_{\mathrm{A}}^{\mathrm{B}} \mathcal{K}(u)u\mathrm{d}u.$
- $\sigma^2_{\mathrm{H},\mathcal{K}} := \mathrm{H}(2\mathrm{H}-1) \int_{\mathrm{A}}^{\mathrm{B}}\int_{\mathrm{A}}^{\mathrm{B}} |u-v|^{2\mathrm{H}-2}\mathcal{K}(u)\mathcal{K}(v)\mathrm{d}u\mathrm{d}v.$

Asymptotic distribution: sketch of proof

Steps:

1. By Inequality (15),

$$\sigma^{-1/(2-\mathrm{H})}(\alpha_\sigma(t)+\zeta_\sigma(t)) \xrightarrow[\sigma\to 0]{\mathbb{L}^2} 0.$$

2. Since $t \in \mathcal{E}_l \cup \mathcal{E}_u \cup \mathcal{E}_{\mathrm{int(C)}}$, $\dot{y}$ is continuous at time t and then

$$\sigma^{-1/(2-\mathrm{H})}(\beta_\sigma(t)+\eta_\sigma(t)) \xrightarrow[\sigma\to 0]{\mathbb{L}^2} \mu(t).$$

3. Since $\dot{\gamma}_\sigma(t)$ is defined by a Wiener integral,

$$\sigma^{-1/(2-\mathrm{H})}\dot{\gamma}_\sigma(t) \xrightarrow[\sigma\to 0]{\Delta} \mathcal{N}(0,\sigma^2_{\mathrm{H},\mathcal{K}}).$$

Thank you for your attention!