Young-trag-2021 : GDR TRAG Young researchers meeting

December 3, 2021
Institut Henri Poincaré
Organised by Florian Bechtold, Adeline Fermanian, David Lee and Rita Nader

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09:00-09:30: Opening

09:30-11:00: Session 1- *(S)PDEs*
Speaker 1: Nimit Rana
"*Random dynamical system generated by the $3D$ Navier-Stokes equation with rough transport noise.*"
Speaker 2: Emanuela Gussetti
"*A pathwise stochastic Landau-Lifshitz-Gilbert equation with application to large deviations.*"
Speaker 3: Cristopher Salvi
"*Neural SPDEs.*"
Speaker 4: Yueh-Sheng Hsu
"*Asymptotic of the smallest eigenvalues of the continuous Anderson Hamiltonian in $d \leq 3$.*"
11:00-11:30: Coffee/tea Break

11:30-13:00: Session 2- *Regularization by noise*
Speaker 1: Lukas Anzeletti
"*Regularization by noise for one-dimensional differential equations with nonnegative distributional drift.*"
Speaker 2: Lucio Galeati
"*Some recent advances on SDEs with fractional noise.*"
Speaker 3: Helena Kremp
"*Weak rough paths type solutions for singular Lévy SDEs.*"
Speaker 4: Lukasz Madry
"*Reflected ODEs with distributional drift.*"

13:00-14:30: Lunch

14:30-15:45: Session 3- *Rough path theory*
Speaker 1: Lucas Broux
"*The Sewing lemma for $0 < \gamma \leq 1$.*"
Speaker 2: William Salkeld
"*Lions calculus and probabilistic rough paths.*"
Speaker 3: Carlo Bellingeri
"*Smooth rough paths.*"
15:45-16:15: Coffee/tea Break

16:15-17:45: Session 4- *Signatures and rough paths for machine learning*
Speaker 1: Leonard Schmitz
"*Attention on signature-type features: beating quadratic costs.*"
Speaker 2: Raphaël Mignot
"*Defining signature barycenters.*"
Speaker 3: Nathan Doumèche
"*Signed Taylor Expansion and applications to machine learning.*"
Speaker 4: Csaba Toth
"*Signature Fourier Features for Scalable Approximation of the Signature Kernel.*"

Session 1- *(S)PDEs*

Speaker 1: Nimit Rana
"*Random dynamical system generated by the $3D$ Navier-Stokes equation with rough transport noise.*"

Abstract: *We consider the Navier-Stokes system in three dimensions perturbed by a transport noise which is sufficiently smooth in space and rough in time. The existence of a weak solution was proved recently, however, as in the deterministic setting the question of uniqueness remains a major open problem. An important feature of systems with uniqueness is the semigroup property satisfied by their solutions. Without uniqueness, this property cannot hold generally. We select a system of solutions satisfying the semigroup property with appropriately shifted rough path. Under suitable assumptions on the driving rough path, we show that the considered Navier-Stokes system generates a measurable random dynamical system.*

Speaker 2: Emanuela Gussetti
"*A pathwise stochastic Landau-Lifshitz-Gilbert equation with application to large deviations.*"

Abstract: *Using a rough path formulation, we investigate existence, uniqueness and regularity for the stochastic Landau-Lifshitz-Gilbert equation with Stratonovich noise on the one dimensional torus. As a main result we show the continuity of the so-called Ito-Lyons map in the energy spaces* $L^\infty(0,T;H^k) \cap L^2(0,T;H^{k+1})$ *for any* $k \geq 1$. *The proof proceeds in two steps. First, based on an energy estimate in the aforementioned space together with a compactness argument we prove existence of a unique solution, implying the continuous dependence in a weaker norm. This is then strengthened in the second step where the continuity in the optimal norm is established through an application of the rough Gronwall lemma. Our approach is direct and does not rely on any transformation formula, which permits to treat multidimensional noise. As an easy consequence we then deduce a Wong-Zakai type result, a large deviation principle for the solution and a support theorem.*

Speaker 3: Cristopher Salvi
"*Neural SPDEs.*"

Abstract: *Stochastic partial differential equations (SPDEs) are the mathematical tool of choice to model dynamical systems evolving under the influence of randomness. By formulating the search for mild solutions of an SPDE as fixed-point problems, we introduce the Neural SPDE model to learn solution operators of SPDEs from partially observed data. Neural SPDEs provide an extension to two classes of physics-inspired neural architectures. On the one hand, it extends the class of neural differential equations (Neural CDE, SDE, RDE) – continuous-time analogues of RNNs – in that it is capable of processing incoming sequential information even when the latter evolves in an infinite dimensional state space. On the other hand, Neural SPDEs extend Neural Operators (NOs) – generalizations of neural networks to model mappings between spaces of functions – in that they can be used to learn solution operators of SPDEs (a.k.a. Ito maps) depending simultaneously on the initial condition and a realization of the driving noise (while there is no natural mechanism allowing to do so with NOs). The Neural SPDE model is resolution-invariant (in that it can be trained efficiently on coarser grids and then deployed on finer grids without sacrificing performance), it may utilize memory-efficient implicit-differentiation-based backpropagation and, once trained, its evaluation is up to* 3 *orders of magnitude faster than traditional SPDE solvers. Through experiments on various semilinear SPDEs, including*

the $2D$ stochastic Navier-Stokes equations, we demonstrate how Neural SPDEs are capable of learning complex spatiotemporal dynamics with better accuracy and using only a modest amount of training data compared to all alternative models.

Speaker 4: Yueh-Sheng Hsu
"*Asymptotic of the smallest eigenvalues of the continuous Anderson Hamiltonian in $d \leq 3$.*"

<u>Abstract</u>: *We consider the continuous Anderson Hamiltonian with white noise potential on $(L/2, L/2)^d$ in dimension $d \leq 3$, and derive the asymptotic of the smallest eigenvalues when L goes to infinity. We show that these eigenvalues go to ∞ at speed $(\log L)^{(1/(2-d/2))}$ and identify the prefactor in terms of the optimal constant of the Gagliardo- Nirenberg inequality. This result was already known in dimensions 1 and 2, but appears to be new in dimension 3. We present some conjectures on the fluctuations of the eigenvalues and on the asymptotic shape of the corresponding eigenfunctions near their localisation centers.*

Session 2- *Regularization by noise*

Speaker 1: Lukas Anzeletti
"*Regularization by noise for one-dimensional differential equations with nonnegative distributional drift.*"

<u>Abstract</u>: *We study existence and uniqueness of solutions to the equation $Xt = b(Xt)dt + dBt$, where b may be distributional and B is a fractional Brownian motion with Hurst parameter $H \leq 1/2$. We are considering the usual probabilistic notion of a solution of an SDE and solutions in a (deterministic) path-by-path sense. In the case of b being in a class of distributions containing nonnegative finite measures, we show existence of path-by-path solutions for $H < \sqrt(2)\breve{}1$ and the existence of a unique strong solution for $H \leq 1/4$. The former will be proven with an Euler scheme using a construction of the nonlinear Young integral in p-variation. The proof of the latter heavily relies on the stochastic sewing lemma.*

Speaker 2: Lucio Galeati
"*Some recent advances on SDEs with fractional noise.*"

<u>Abstract</u>: *In recent years, there has been a lot of interest in regularization by noise for SDEs driven by fractional Brownian motion of parameter $H \in (0, 1)$, with first results going back to Nualart, Ouknine (2002) and Catellier, Gubinelli (2016). In particular, strong well-posedness is known for drifts of regularity $\alpha > 1 - 1/(2H)$. In this talk I will present some novel results on the topic, including generalizations of the above in the regime $H \in (1, \infty)$, stability estimates for SDEs driven by different drifts and solvability of McKean-Vlasov equations.*

Speaker 3: Helena Kremp
"*Weak rough paths type solutions for singular Lévy SDEs.*"

<u>Abstract</u>: *Since the works by Delarue, Diel and Cannizzaro, Chouk (in the Brownian noise setting), and our previous work, the existence and uniqueness of solutions to the martingale problem associated to multidimensional SDEs with additive alpha-stable Lévy noise for $\alpha in (1, 2]$ and rough Besov drift of regularity $\beta > (2 - 2\alpha)/3$ is known. Motivated by the equivalence of probabilistic weak solutions to SDEs with bounded, measurable drift and solutions to the martingale problem, we define a (non-canonical)*

weak solution concept for singular Lévy diffusions. We then prove equivalence to the martingale solution in both the Young ($\beta > (1-\alpha)/2$), as well as in the rough regime ($\beta > (2-2\alpha)/3$). This turns out to be highly non-trivial in the rough case and forces us to define certain rough stochastic sewing integrals involved. In particular, we show that the canonical weak solution concept (introduced also by Athreya, Butkovsky, Mytnik in the Young case), which is well-posed in the Young case, yields non-uniqueness of solutions in the rough case.

Speaker 4: Lukasz Madry
"*Reflected ODEs with distributional drift.*"

Abstract: *We consider differential equations with reflection, i.e. which are constrained to remain in a given domain. We obtain "regularization by noise" results in this context, namely we show that adding a (fractional) noise term allows to restore well-posedness for equations driven by singular (possibly distributional) drift vector fields. Our proof follows the methods developed by Catellier-Gubinelli based on nonlinear Young integration, combined with a Lipschitz property for the Skorokhod map due to Falkowski-Slominski. In order to obtain well-posedness under optimal regularity assumptions on the vector fields (i.e. in the regime where Girsanov theorem is needed), we show that suitable perturbations of fBm (such as reflected fBm) have similar pathwise regularizing properties as fBm.*

Session 3- *Rough path theory*

Speaker 1: Lucas Broux
"*The Sewing lemma for* $0 < \gamma \leq 1$."

Abstract: *We establish a Sewing lemma in the regime $\gamma \in (0,1]$, constructing a Sewing map which is neither unique nor canonical, but which is nonetheless continuous with respect to the standard norms. Two immediate corollaries follow, which hold on any commutative graded connected locally finite Hopf algebra: a simple constructive proof of the Lyons-Victoir extension theorem which associates to a Hölder path a rough path, with the additional result that this map can be made continuous; the bicontinuity of a transitive free action of a space of Hölder functions on the set of Rough Paths.*

Speaker 2: William Salkeld
"*Lions calculus and probabilistic rough paths.*"

Abstract: *In this talk, I will explain some of the foundation results for a new regularity structure developed to study interactive systems of equations and their mean-field limits. At the heart of this solution theory is a Taylor expansion using the so called Lions measure derivative. This quantifies infinitesimal perturbations of probability measures induced by infinitesimal variations in a linear space of random variable.I will explore how basic properties of Lions derivatives evolve into the structures of a coupled Hopf algebra and use this to construct a probabilistic rough path. This allows us to formulate solutions of McKean-Vlasov equations in a pathwise setting while simultaneously capturing the dynamics of thesolution law.*

Speaker 3: Carlo Bellingeri
"*Smooth rough paths.*"

Abstract: *We introduce the class of "smooth rough paths" and study their main properties from a*

geometric perspective, both in the geometric, quasi-geometric and Hopf algebraic settings. The key-notion to analyse them turns out to be the diagonal derivative, i.e. the derivative of the path in the Lie Group of characters pulled back to the underlying Lie algebra. With that concept, we obtain an analogue of Lyons extension theorem based on a purely algebraic condition. We furthermore introduce a canonical sum of smooth rough paths, an operation that does not exist between two genuine rough paths.

Session 4- *Signatures and rough paths for machine learning*

Speaker 1: Leonard Schmitz
"*Attention on signature-type features: beating quadratic costs.*"

<u>Abstract</u>: *The attention mechanism is a fundamental building block for sequence-to-sequence tasks from machine learning. We show that its high time complexity can be reduced by a magnitude for low-rank decompositions of the input data. The signature provides such low-rank data when used as a feature of sequences. We present experimental results of an efficient matching mechanism for subsequences which combines attention and the signature method.*

Speaker 2: Raphaël Mignot
"*Defining signature barycenters.*"

<u>Abstract</u>: *The analysis of time series through the signature transformation of the data has now been used extensively and has shown to perform well in various Machine Learning contexts. Designing a notion of barycenter in the signature space (Lie group) would allow the use of the signature transform in ubiquitous strategies in Data Science such as the Principal Component Analysis (PCA) for data compression or k-means for clustering.*

Speaker 3: Nathan Doumèche
"*Signed Taylor Expansion and applications to machine learning.*"

<u>Abstract</u>: *A new framework has emerged among the Machine Learning community which consists in linking neural networks to differential equations. Indeed, several architectures, such as ResNet, RNN, VAE, GRU and LSTM, can be represented by differential equations. From learning the parameters of these architectures, we therefore only have to learn the coefficients of the corresponding differential equations. This talk aims at presenting a Taylor expansion, which we will call the Signed Taylor Expansion, that can be used to solve these differential equations. We will show that, thanks to a mathematical object called the signature, these differential equations are in fact linear operations. All in all, the problem only comes to learning parameters of linear opperations applied to the signature of the inputs. These interpretation of the signature as a kernel links our problem.*

Speaker 4: Csaba Toth
"*Signature Fourier Features for Scalable Approximation of the Signature Kernel.*"

<u>Abstract</u>: *Signature features, originating from rough paths theory, have shown strong performance for modelling functions of sequences in data science applications. A key idea behind learning with signatures in the kernel framework is a kernel trick, which allows to 1) circumvent the dimensionality bottleneck, 2) compose signature kernels with static kernels, making the computations feasible even if the associated static feature map is infinite-dimensional. The cost of this is a quadratic complexity*

in sequence length, which stems from performing all pairwise comparisons between the entries of two input sequences, limiting the scalability of the approach. On the other hand, the Random Fourier Feature approximation is a classic tool for increasing the scalability of stationary kernels by making use of the Fourier duality between stationary kernels and non-negative finite measures. We overcome the quadratic complexity bottleneck of signature kernels while maintaining the previous benefits by introducing Signature Fourier Features, an extension of the Fourier feature framework that respects the tensor-product structure of signatures. These random features provide an unbiased approximation to the (lifted) signature kernel, and converge fast in probability as the embedding dimension of the random feature map increases.